

Lecture 18

Iterative solution of linear systems*

Newton refinement

Conjugate gradient method

Review of Part II

Methods and Formulas

Basic Matrix Theory:

Identity matrix: $AI = A$, $IA = A$, and $I\mathbf{v} = \mathbf{v}$

Inverse matrix: $AA^{-1} = I$ and $A^{-1}A = I$

Norm of a matrix: $\|A\| \equiv \max_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|$

A matrix may be singular or nonsingular. See Lecture 10.

Solving Process:

Gaussian Elimination produces LU decomposition

Row Pivoting

Back Substitution

Condition number:

$$\text{cond}(A) \equiv \max \left(\frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}} \right) = \max \left(\frac{\text{Relative error of output}}{\text{Relative error of inputs}} \right).$$

A big condition number is bad; in engineering it usually results from poor design.

LU factorization:

$$PA = LU.$$

Solving steps:

Multiply by P: $\mathbf{d} = P\mathbf{b}$

Forwardsolve: $L\mathbf{y} = \mathbf{d}$

Backsolve: $U\mathbf{x} = \mathbf{y}$

Eigenvalues and eigenvectors:

A nonzero vector \mathbf{v} is an eigenvector and a number λ is its eigenvalue if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Characteristic equation: $\det(A - \lambda I) = 0$

Equation of the eigenvector: $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Residual for an approximate eigenvector-eigenvalue pair: $r = \|A\mathbf{v} - \lambda\mathbf{v}\|$

Complex eigenvalues:

Occur in conjugate pairs: $\lambda_{1,2} = \alpha \pm i\beta$

and eigenvectors must also come in conjugate pairs: $\mathbf{w} = \mathbf{u} \pm i\mathbf{v}$.

Vibrational modes:

Eigenvalues are frequencies squared. Eigenvectors represent modes.

Power Method:

- Repeatedly multiply \mathbf{x} by A and divide by the element with the largest absolute value.
- The element of largest absolute value converges to largest absolute eigenvalue.
- The vector converges to the corresponding eigenvector.
- Convergence assured for a real symmetric matrix, but not for an arbitrary matrix, which may not have real eigenvalues at all.

Inverse Power Method:

- Apply power method to A^{-1} .
- Use solving rather than the inverse.
- If λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue for A^{-1} .
- The eigenvectors for A and A^{-1} are the same.

Symmetric and Positive definite:

- Symmetric: $A = A'$.
- If A is symmetric its eigenvalues are real.
- Positive definite: $A\mathbf{x} \cdot \mathbf{x} > 0$.
- If A is positive definite, then its eigenvalues are positive.

QR method:

- Transform A into H the Hessenberg form of A .
- Decompose H in QR .
- Multiply Q and R together in reverse order to form a new H .
- Repeat
- The diagonal of H will converge to the eigenvalues of A .

Matlab

Matrix arithmetic:

$A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ -1 & -1 & 5 & 4 \\ 0 & 1 & -9 & 0 \end{bmatrix}$ Manually enter a matrix.
 $u = [1 \ 2 \ 3 \ 4]'$
 $A*u$
 $B = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix}$
 $B*A$ multiply B times A .
 $2*A$ multiply a matrix by a scalar.
 $A + A$ add matrices.
 $A + 3$ add 3 to every entry of a matrix.
 $B.*B$ component-wise multiplication.
 $B.^3$ component-wise exponentiation.

Special matrices:

$I = \text{eye}(3)$ identity matrix
 $D = \text{ones}(5,5)$
 $O = \text{zeros}(10,10)$
 $C = \text{rand}(5,5)$ random matrix with uniform distribution in $[0,1]$.
 $C = \text{randn}(5,5)$ random matrix with normal distribution.
 $\text{hilb}(6)$
 $\text{pascal}(5)$

General matrix commands:

$\text{size}(C)$ gives the dimensions ($m \times n$) of A .
 $\text{norm}(C)$ gives the norm of the matrix.
 $\text{det}(C)$ the determinant of the matrix.
 $\text{max}(C)$ the maximum of each row.
 $\text{min}(C)$ the minimum in each row.
 $\text{sum}(C)$ sums each row.
 $\text{mean}(C)$ the average of each row.
 $\text{diag}(C)$ just the diagonal elements.
 $\text{inv}(C)$ inverse of the matrix.
 C' transpose of the matrix.

Matrix decompositions:

$[L \ U \ P] = \text{lu}(C)$
 $[Q \ R] = \text{qr}(C)$
 $H = \text{hess}(C)$ transform into a Hessian tri-diagonal matrix, which has the same eigenvalues as A .

Part III

Functions and Data

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