

Lecture 15

An Application of Eigenvectors: Vibrational Modes and Frequencies

One application of eigenvalues and eigenvectors is in the analysis of vibration problems. A simple nontrivial vibration problem is the motion of two objects with equal masses m attached to each other and fixed outer walls by equal springs with spring constants k , as shown in Figure 15.1.

Let x_1 denote the displacement of the first mass and x_2 the displacement of the second, and note the displacement of the walls is zero. Each mass experiences forces from the adjacent springs proportional to the stretch or compression of the spring. Ignoring any friction, Newton's law of motion $ma = F$, leads to

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - 0) + k(x_2 - x_1) &= -2kx_1 + kx_2 & \text{and} \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(0 - x_2) &= kx_1 - 2kx_2 & . \end{aligned} \quad (15.1)$$

Dividing both sides by m we can write these equations in matrix form

$$\ddot{\mathbf{x}} = -A\mathbf{x}, \quad (15.2)$$

where

$$A = \frac{k}{m}B = \frac{k}{m} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (15.3)$$

For this type of equation, the general solution is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 \sin\left(\sqrt{\frac{k\lambda_1}{m}} t + \phi_1\right) + c_2 \mathbf{v}_2 \sin\left(\sqrt{\frac{k\lambda_2}{m}} t + \phi_2\right) \quad (15.4)$$

where λ_1 and λ_2 are eigenvalues of B with corresponding eigenvectors \mathbf{v}_1 and \mathbf{v}_2 . One can check that this is a solution by substituting it into the equation (15.2).

We can interpret the eigenvalues as the squares of the frequencies of oscillation. We can find the eigenvalues and eigenvectors of B using Matlab:

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B = [2 -1 ; -1 2]
[v e] = eig(B)
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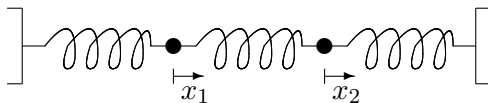


Figure 15.1: Two equal masses attached to each other and fixed walls by equal springs.

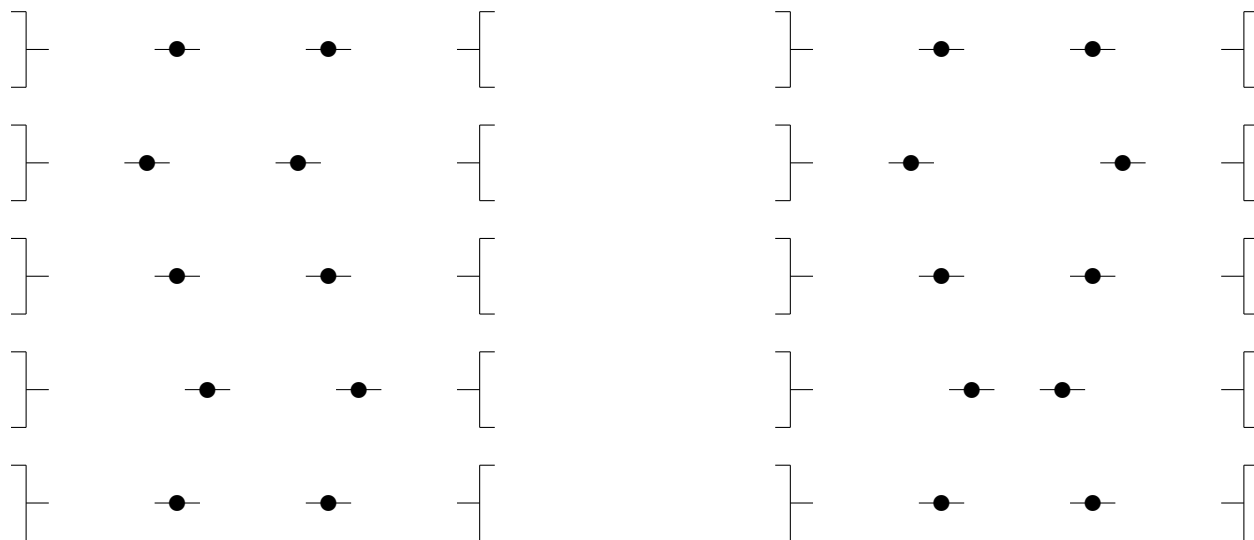


Figure 15.2: Two vibrational modes of a simple oscillating system. In the left mode the weights move together and in the right mode they move opposite. Note that the two modes actually move at different speeds.

This should produce a matrix \mathbf{v} whose columns are the eigenvectors of B and a diagonal matrix e whose entries are the eigenvalues of B . In the first eigenvector, \mathbf{v}_1 , the two entries are equal. This represents the mode of oscillation where the two masses move in sync with each other. The second eigenvector, \mathbf{v}_2 , has the same entries but opposite signs. This represents the mode where the two masses oscillate in anti-synchronization. Notice that the frequency for anti-sync motion is $\sqrt{3}$ times that of synchronous motion.

Which of the two modes is the most dangerous for a structure or machine? It is the one with the *lowest frequency* because that mode can have the largest displacement. Sometimes this mode is called the *fundamental mode*.

To get the frequencies for the matrix $A = k/mB$, notice that if \mathbf{v}_i is one of the eigenvectors for B then

$$A\mathbf{v}_i = \frac{k}{m}B\mathbf{v}_i = \frac{k}{m}\lambda_i\mathbf{v}_i.$$

Thus we can conclude that A has the same eigenvectors as B , but the eigenvalues are multiplied by the factor k/m . Thus the two frequencies are

$$\sqrt{\frac{k}{m}} \quad \text{and} \quad \sqrt{\frac{3k}{m}}.$$

We can do the same for three equal masses. The corresponding matrix B would be

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the eigenvectors and eigenvalues as above. There are three different modes. Interpret them from the eigenvectors.

Exercises

- 15.1 Find the frequencies and modes for 4 equal masses with equal springs with $k = 1$. Interpret the modes.
- 15.2 Find the frequencies and modes for non-identical masses with equal springs with $k = 3$ in the following cases. How does unequal masses affect the modes?
- (a) Two masses with $m_1 = 1$ and $m_2 = 2$.
 - (b) Three masses with $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$.