Trigonometric Identities and Sums of Separable Functions

Martin J. Mohlenkamp* Lucas Monzón†

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For what value(s) of α, β, γ does the equality

$$\begin{aligned}\sin(x+y+z) &= \sin(x)\frac{\sin(y+\beta-\alpha)}{\sin(\beta-\alpha)}\frac{\sin(z+\gamma-\alpha)}{\sin(\gamma-\alpha)} \\ &+ \frac{\sin(x+\alpha-\beta)}{\sin(\alpha-\beta)}\sin(y)\frac{\sin(z+\gamma-\beta)}{\sin(\gamma-\beta)} \\ &+ \frac{\sin(x+\alpha-\gamma)}{\sin(\alpha-\gamma)}\frac{\sin(y+\beta-\gamma)}{\sin(\beta-\gamma)}\sin(z)\end{aligned} \tag{1}$$

hold for all values of x, y, and z?

Motivation

Modern computers have made commonplace many calculations that were impossible to imagine a few years ago. Still, when you face a problem with a high physical dimension, you immediately encounter the *Curse of Dimensionality* [1, p.94]. This curse is that the amount of computing power that you need grows exponentially with the dimension. The simplest manifestation of this curse appears when you try to represent a function by its sample values on a grid. If a function of one variable requires N samples, then an analogous function of n variables will need a grid of N^n samples. Thus, even relatively small problems in high dimensions are still unreasonably expensive.

A method has been proposed in [2] to address this problem, based on approximating a function by a sum of separable functions:

$$f(x_1,\ldots,x_n) \approx \sum_{j=1}^{r} \phi_1^j(x_1)\phi_2^j(x_2)\cdots\phi_n^j(x_n). \tag{2}$$

This representation would require only $r\,n\,N$ samples, so if the approximation can be made sufficiently accurate while keeping the separation rank r small, we can bypass the curse.

We describe here a particular test of (2), when the "straightforward" approximation is exact but has very large separation rank. Although it may not be directly useful in applications, the result of this test is surprising, positive, and, we believe, cute. It illustrates a richness of structure that invites future study. Other mechanisms that allow representations of the form (2) with low separation rank are described in [2].

*Department of Mathematics, Ohio University, 321 Morton Hall, Athens OH 45701; mjm@math.ohiou.edu. This material is based in part upon work supported by the National Science Foundation under Grant No. DMS-9902365.

†Department of Applied Mathematics, University of Colorado, 526 UCB, Boulder CO 80309-0526; monzon@colorado.edu. This material is based in part upon work supported by University of Virginia subcontract MDA-972-00-1-0016.

A Test Function

Our test function is sine of the sum of n variables, $\sin(\sum_{j=1}^{n} x_j)$, which is a wave oriented in the "diagonal" direction in n-dimensional space. One could use complex exponentials to express it as the sum of two separable functions,

$$\sin(\sum_{j=1}^{n} x_j) = \frac{1}{2i}\prod_{j=1}^{n} e^{ix_j} - \frac{1}{2i}\prod_{j=1}^{n} e^{-ix_j},$$

but in our test we only allowed real functions.

You can use ordinary trigonometric identities to find such a representation. When $n = 2$ we have

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y), \tag{3}$$

which expresses $\sin(x + y)$ as a sum of two separable functions. When $n = 3$ we have

$$\begin{aligned} \sin(x + y + z) &= \sin(x)\cos(y)\cos(z) + \cos(x)\cos(y)\sin(z) \\ &+ \cos(x)\sin(y)\cos(z) - \sin(x)\sin(y)\sin(z), \end{aligned} \tag{4}$$

which uses four terms. The drawback to this approach is that, for n variables, the number of terms is 2^{n-1}. This exponential growth in the number of terms negates the benefit of using the form (2). Indeed, if this really is the minimal number of terms needed, then the entire approach is doomed.

We then asked what the minimal number of terms is, and our program replied "n" and produced graphs such as that shown in Figure 1(right). After some investigation, we determined the trigonometric identity that our program had uncovered. What is most remarkable is that the program was numerical, not symbolic, and so uncovered a trigonometric identity without even knowing it was doing trigonometry!

Any representation of a function of a sum of n variables will have $n-1$ free parameters, since one can include n shifts $x_j \to x_j + a_j$ and one linear constraint $\sum_{j=1}^{n} a_j = 0$ and have $\sum_{j=1}^{n} x_j + a_j = \sum_{j=1}^{n} x_j$. The identity that we present in Theorem 2 has $n - 1$ additional independent parameters, which play a structural role in our representation. When $n = 3$, it provides an answer for our opening teaser. The identity (1) holds for arbitrary α, β, and γ, as long as $\sin(\alpha - \beta) \neq 0$, $\sin(\alpha - \gamma) \neq 0$, and $\sin(\beta - \gamma) \neq 0$. Since these three parameters occur only as differences, only two of them are independent. One can introduce two additional parameters as phase shifts to make versions of (1) with different symmetries.

The Identity

Lemma 1 *The function $s(x) = \sin(x)$ satisfies the equation*

$$s(A + B) = \frac{s(A)s(B + \beta - \alpha)}{s(\beta - \alpha)} + \frac{s(A + \alpha - \beta)s(B)}{s(\alpha - \beta)} \tag{5}$$

for all values of A, B, α, and β such that $s(\alpha - \beta) \neq 0$ (and $s(\beta - \alpha) \neq 0$).

Proof. With the notation $c(x) = \cos(x)$ and $\gamma = \beta - \alpha$, partially expand the right-hand side using the usual trigonometric identity (3) to obtain

$$\frac{s(A)}{s(\gamma)}(s(B)c(\gamma) + c(B)s(\gamma)) + \frac{s(B)}{s(-\gamma)}(s(A)c(-\gamma) + c(A)s(-\gamma))$$

Multiplying out and using that $s(x)$ is odd and $c(x)$ is even, all terms cancel except for $s(A)c(B) + c(A)s(B)$, which we recognize as $s(A + B)$. $\square$

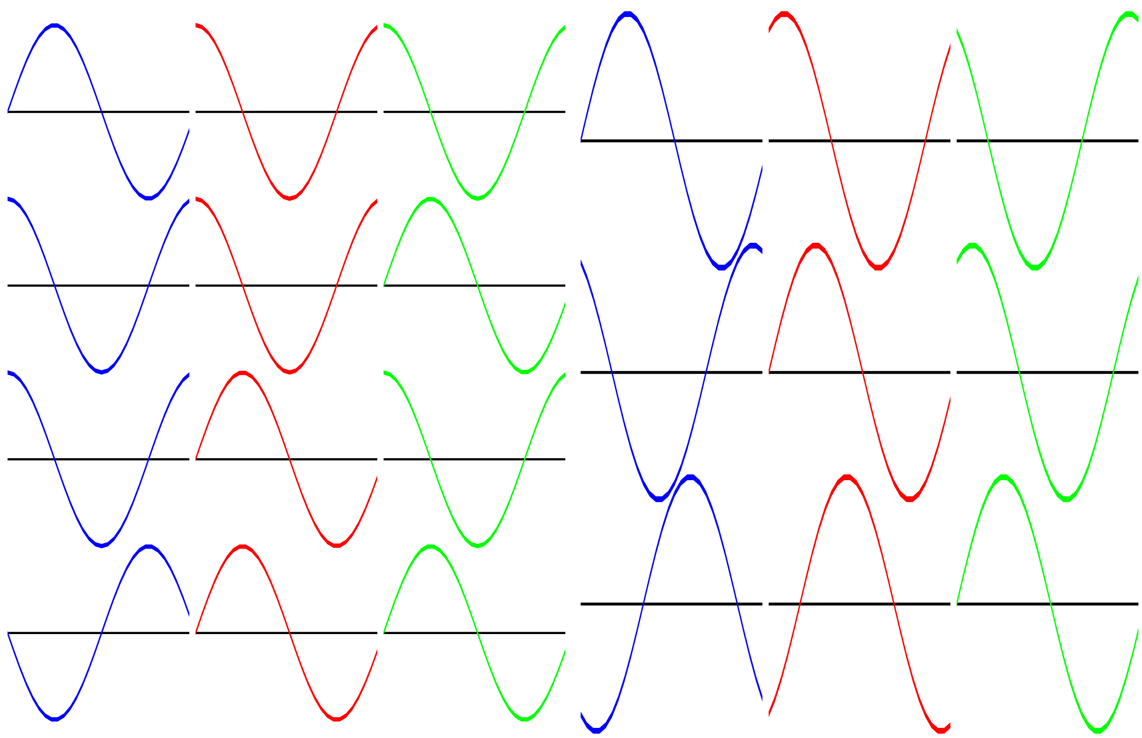

Figure 1: **left:** Graphical separated representation of $\sin(x+y+z)$ using the usual trigonometric identity (4). Each of the four rows gives the factors of a separable function. For example, the first row corresponds to $\sin(x)\cos(y)\cos(z)$. The separable functions from each row are then added. **right:** Graphical separated representation of $\sin(x+y+z)$ using (1) with $\alpha = 0$, $\beta = \pi/3$, and $\gamma = 2\pi/3$. The amplitude has been equidistributed.

Theorem 2 *Any function $s(x)$ that satisfies (5) also satisfies*

$$s\left(\sum_{j=1}^{n} x_j\right) = \sum_{j=1}^{n} s(x_j) \prod_{k=1,k\neq j}^{n} \frac{s(x_k+\alpha_k-\alpha_j)}{s(\alpha_k-\alpha_j)}. \tag{6}$$

for all choices of $\{\alpha_j\}$ such that $s(\alpha_k-\alpha_j)\neq 0$ for all $j\neq k$.

The proof is by induction and is given in the Appendix. We can generate a more general form by introducing n shifts a_j:

$$s\left(\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} s(x_j+a_j) \prod_{k=1,k\neq j}^{n} \frac{s(x_k+a_k+\alpha_k-\alpha_j)}{s(\alpha_k-\alpha_j)}.$$

By choosing different ways to satisfy the linear constraint $\sum_{j=1}^{n} a_j = 0$, we can produce a variety of identities similar to (6) without modifying the parameters α_j, which are the structural elements of our representation. Note that in the set $\{\alpha_k-\alpha_j\}_{k\neq j}$ only $n-1$ parameters are linearly independent, say $\{\alpha_1-\alpha_j\}_{j=2}^{n}$.

Other Functions that Satisfy the Same Identity

Because of Lemma 1 and Theorem 2, we know that $\sin(\sum_{j=1}^{n} x_j)$ is exactly separated with separation rank n. Moreover, this function is peculiar in that $\sin(\cdot)$ is the only function used in the separated representation. We now consider the problem of finding other functions $s(x)$ satisfying (6). Since the general case (6) is equivalent to the $n=2$ case, it is enough to describe all functions that satisfy (5).

Lemma 3 *The function $s(x)=x$ satisfies the identity (5).*

This and the following lemma may be verified directly.

Lemma 4 *If $s(x)$ satisfies (5), then so does*

$$a\exp(bx)s(cx)$$

for all complex $a\neq 0$, b, and $c\neq 0$.

Starting with our two basic functions $\sin(x)$ and x, we can use Lemma 4 to construct other functions that satisfy (5), and then ask if we have missed any others. We only wish to consider reasonably nice functions. The technical condition that we need for the proof of the following theorem, given in the Appendix, is that $s(x)$ be meromorphic.

Theorem 5 *A meromorphic function $s(x)$ satisfies (5) if and only if*

$$s(x) = a\exp(bx)x \qquad \textit{or} \qquad s(x) = a\exp(bx)\sin(cx)$$

for some complex constants $a\neq 0$, b, and $c\neq 0$.

Extensions and Relationships with Other Identities

If in Theorem 2 we set $x_j=\alpha$ for all j, we obtain the following corollary.

Corollary 6 *Under the same conditions as in Theorem 2,*

$$\frac{s(n\alpha)}{s(\alpha)} = \sum_{j=1}^{n} \prod_{k=1,k\neq j}^{n} \frac{s(\alpha+\alpha_k-\alpha_j)}{s(\alpha_k-\alpha_j)}.$$

When $s(x) = \sin(x)$, this result is presented in [4]. For a proof using Lagrangian interpolation see [5, page 272]. The approach of [4] and [5], however, does not produce the general results of Theorems 2 and 5. Conversely, our results can only be used to derive a few of the identities listed in [5, Section 2.4.5.3].

The situation is different if we consider Milne's identity [6, 3]

$$1 - \prod_{j=1}^{n} y_j = \sum_{j=1}^{n} (1 - y_j) \prod_{k=1, k\neq j}^{n} \frac{1 - y_k\, \theta_k/\theta_j}{1 - \theta_k/\theta_j}. \tag{7}$$

We can obtain another proof of this identity by setting $s(x) = 1 - \exp(x)$, $\alpha_j = \ln \theta_j$, and $x_j = \ln y_j$ in (6). Conversely, (6) for $s(x) = \sin(x)$ can be obtained by setting $y_j = \exp(-2ix_j)$ and $\theta_j = \exp(-2i\alpha_j)$ in Milne's identity, and then multiplying by $\exp(i \sum_{j=1}^{n} x_j)/2i$.

A "multiplicative" version of the identities that we have discussed can be derived by generalizing this observation. Simply note that the identity

$$f(CD) = \frac{f(C) f(D\phi/\theta)}{f(\phi/\theta)} + \frac{f(C\theta/\phi) f(D)}{f(\theta/\phi)}$$

is equivalent to (5) with the substitutions $C = \exp(A)$, $D = \exp(B)$, $\theta = \exp(\alpha)$, $\phi = \exp(\beta)$, and $s(x) = f(\exp(x))$. Similarly, (6) is equivalent to

$$f(\prod_{j=1}^{n} y_j) = \sum_{j=1}^{n} f(y_j) \prod_{k=1, k\neq j}^{n} \frac{f(y_k \theta_k/\theta_j)}{f(\theta_k/\theta_j)}. \tag{8}$$

In analogy to Lemma 4, from the particular solutions $f(x) = \ln(x)$ and $f(x) = 1 - x$ to (8) we can generate other solutions to (8), namely

$$a x^b f(x^c)$$

for constants a, b, and c. In this way we obtain a generalization of Milne's identity.

Remarks and Conclusions

It is easy to extend our results to find similar identities for $f(\sum_{j=1}^{n} x_j)$, where $f(x)$ could be $\cos(x)$, $\cos^2(x)$, or $\sin^2(x)$, for example.

We also tested the function of six variables $\sin(u+v+w)\sin(x+y+z)$. Using (1) on each factor and then multiplying out yields a representation of the form (2) with 9 terms, but our program found a representation with 8 terms. After considerable effort, we have still not been able to find the formula analogous to (6) for this case.

A survey on the problem of exact separated representations is the book [7] by Rassias and Šimša. As they pointed out in Problem 4 of page 158, to find a minimal rank representation for a separated representation is still an open problem. We believe that our Theorems 2 and 5 are an example of such minimal representations.

Lemma 1 can be proven geometrically, in a way similar to the geometric proof of the usual identity (3). We have not been able to find a geometric interpretation of (6).

Appendix: Proofs

Proof of Theorem 2.

The case $n = 2$ is Lemma 1 with $A = x_1$, $B = x_2$, $\alpha = \alpha_1$, and $\beta = \alpha_2$. The proof will be by induction in n, so we assume (6) has been proven for $n - 1$. We will use (5) to separate out the variable x_n, then cancel like terms and reduce the n case to the $n - 1$ case.

First expand the left-hand side of (6) using (5) with $A = \sum_{j=1}^{n-1} x_j$, $B = x_n$, $\alpha = \alpha_{n-1}$, and $\beta = \alpha_n$ to obtain

$$s\left(\sum_{j=1}^{n-1} x_j\right) \frac{s(x_n + \alpha_n - \alpha_{n-1})}{s(\alpha_n - \alpha_{n-1})} + s\left(\sum_{j=1}^{n-1} x_j + \alpha_{n-1} - \alpha_n\right) \frac{s(x_n)}{s(\alpha_{n-1} - \alpha_n)}. \tag{9}$$

On the right-hand side of (6), first separate off the $j = n$ term in the sum. When $j \neq n$, we expand the $k = n$ term in the product using (5) with $A = \alpha_n - \alpha_j$, $B = x_n$, $\alpha = \alpha_{n-1}$, and $\beta = \alpha_n$. Explicitly, the $k = n$ term is

$$\begin{aligned}
\frac{s(x_n + \alpha_n - \alpha_j)}{s(\alpha_n - \alpha_j)} &= \frac{1}{s(\alpha_n - \alpha_j)} \left(\frac{s(\alpha_n - \alpha_j) s(x_n + \alpha_n - \alpha_{n-1})}{s(\alpha_n - \alpha_{n-1})} + \frac{s(\alpha_{n-1} - \alpha_j) s(x_n)}{s(\alpha_{n-1} - \alpha_n)} \right) \\
&= \frac{s(x_n + \alpha_n - \alpha_{n-1})}{s(\alpha_n - \alpha_{n-1})} + \left(\frac{s(\alpha_{n-1} - \alpha_j)}{s(\alpha_n - \alpha_j)} \right) \left(\frac{s(x_n)}{s(\alpha_{n-1} - \alpha_n)} \right).
\end{aligned}$$

Note that the first term does not depend on j, and that when $j = n - 1$ the second term is absent. Combining these expansions, we can express the right-hand side of (6) as

$$\begin{aligned}
& \left(\sum_{j=1}^{n-1} s(x_j) \prod_{k=1, k\neq j}^{n-1} \frac{s(x_k + \alpha_k - \alpha_j)}{s(\alpha_k - \alpha_j)} \right) \frac{s(x_n + \alpha_n - \alpha_{n-1})}{s(\alpha_n - \alpha_{n-1})} \\
+ & \left(\sum_{j=1}^{n-2} s(x_j) \frac{s(\alpha_{n-1} - \alpha_j)}{s(\alpha_n - \alpha_j)} \prod_{k=1, k\neq j}^{n-1} \frac{s(x_k + \alpha_k - \alpha_j)}{s(\alpha_k - \alpha_j)} \right) \frac{s(x_n)}{s(\alpha_{n-1} - \alpha_n)} \\
+ & \quad s(x_n) \prod_{k=1}^{n-1} \frac{s(x_k + \alpha_k - \alpha_n)}{s(\alpha_k - \alpha_n)}.
\end{aligned} \tag{10}$$

Now compare our expansions (9) and (10) of the two sides of (6). Using the induction hypothesis at $n - 1$, we can see that the first terms in (9) and (10) are equal, and so cancel. The remaining terms all have a factor of $s(x_n)$ in the numerator and $s(\alpha_{n-1} - \alpha_n)$ in the denominator, which we can also cancel. Thus we have reduced the proof to showing that

$$\begin{aligned}
s\left(\sum_{j=1}^{n-1} x_j + \alpha_{n-1} - \alpha_n\right) &= \sum_{j=1}^{n-2} s(x_j) \frac{s(\alpha_{n-1} - \alpha_j)}{s(\alpha_n - \alpha_j)} \prod_{k=1, k\neq j}^{n-1} \frac{s(x_k + \alpha_x - \alpha_j)}{s(\alpha_k - \alpha_j)} \\
&+ \quad s(x_{n-1} + \alpha_{n-1} - \alpha_n) \prod_{k=1}^{n-2} \frac{s(x_k + \alpha_k - \alpha_n)}{s(\alpha_k - \alpha_n)}.
\end{aligned}$$

Now make the substitutions $\tilde{x}_{n-1} = x_{n-1} + \alpha_{n-1} - \alpha_n$ and $\tilde{\alpha}_{n-1} = \alpha_n$ and rearrange to obtain

$$\begin{aligned}
s\left(\sum_{j=1}^{n-2} x_j + \tilde{x}_{n-1}\right) &= \sum_{j=1}^{n-2} s(x_j) \frac{s(\tilde{x}_{n-1} + \tilde{\alpha}_{n-1} - \alpha_j)}{s(\tilde{\alpha}_{n-1} - \alpha_j)} \prod_{k=1, k\neq j}^{n-2} \frac{s(x_k + \alpha_k - \alpha_j)}{s(\alpha_k - \alpha_j)} \\
&+ \quad s(\tilde{x}_{n-1}) \prod_{k=1}^{n-2} \frac{s(x_k + \alpha_k - \tilde{\alpha}_{n-1})}{s(\alpha_k - \tilde{\alpha}_{n-1})}.
\end{aligned}$$

We recognize this equation as the $n - 1$ case of (6), which is true by the induction hypothesis. □

The proof of Theorem 5 depends on two lemmas.

Lemma 7 *If a meromorphic function $s(x)$ satisfies* (5)*, then there exists a complex constant b such that $\exp(-bx)s(x)$ is an odd function.*

Lemma 8 *An odd meromorphic function $s(x)$ satisfies* (5) *if and only if*

$$s(x) = ax \qquad \textit{or} \qquad s(x) = a\sin(cx)$$

for some complex constants $a \neq 0$ and $c \neq 0$.

Proof of Theorem 5*, given Lemmas* 7 *and* 8*.*
We have already shown that these functions satisfy (5), so we need only show there are no more solutions. We now assume $s(x)$ satisfies (5) and will deduce its properties.
Using Lemma 7, we know that $h(x) = \exp(-bx)s(x)$ is an odd function. By Lemma 4, $h(x)$ also satisfies (5). Then, by Lemma 8, $h(x)$ is either ax or $a\sin(cx)$, so $s(x) = a\exp(bx)x$ or $s(x) = a\exp(bx)\sin(x)$, which completes the proof. □

The proofs of Lemmas 7 and 8 use the fact that $s(0) = 0$. By setting $\beta - \alpha = A$ in (5) and subtracting $s(A + B)$ from both sides we obtain

$$0 = \frac{s(0)s(B)}{s(-A)},$$

valid for all A such that $s(-A) \neq 0$ and for all B. Choosing B such that $s(B) \neq 0$ implies that $s(0) = 0$.

Proof of Lemma 7*.*
We define the auxiliary meromorphic function

$$F(x) = -\frac{s(x)}{s(-x)}, \tag{11}$$

which cannot be identically zero, and show that it satisfies the functional equation

$$F(x + w) = F(x)F(w). \tag{12}$$

This functional equation is only satisfied by exponentials, so we can conclude that $F(x) = \exp(2bx)$ for some constant b. Rewriting this condition in terms of s, we have $\exp(-bx)s(x) = -\exp(bx)s(-x)$, which is what we are trying to show.
To show (12), we substitute in (11) and manipulate to form the equivalent equation

$$0 = \frac{s(x)s(w)}{s(x+w)} + \frac{s(-w)s(-x)}{s(-x-w)}. \tag{13}$$

Using (5) with $A = x$, $B = -x$, $\alpha = -x$, and $\beta = w$, we conclude that the right-hand side of (13) is equal to $s(0) = 0$. □

Proof of Lemma 8*.*
Taking a derivative with respect to A in (5), using the fact that s is odd, and setting $A = -\alpha$, $B = \alpha$ and $\beta = -\alpha$, we obtain

$$s'(0)s(2\alpha) = 2s(\alpha)s'(\alpha).$$

Thus, $s'(0) \neq 0$, and because of the invariance with respect to multiplication by constants, we can assume $s'(0) = 1$. We have the system

$$\begin{cases} s'(0) &= 1 \\ s(2\alpha) &= (s^2(\alpha))'. \end{cases}$$

Since $s(0) = 0$, we know that s is analytic around zero. We can write $s(z) = \sum_{k=0}^{\infty} a_k z^{2k+1}$ and use the previous conditions to obtain a recurrence for the sequence a_n,

$$\begin{cases} a_0 &= 1 \\ 2^{2n+1} a_n &= (2n+2) \sum_{k=0}^{n} a_{n-k} a_k. \end{cases} \tag{14}$$

The value of a_n for $n > 1$ is uniquely determined by the value of a_1, which is arbitrary. Setting $\lambda = 6a_1$ we claim

$$a_n = \frac{\lambda^n}{(2n+1)!}. \tag{15}$$

When $\lambda = 0$ we have $s(x) = x$ and when $\lambda \neq 0$ we have $s(x) = \sin(\lambda x)$ and the Lemma follows . We prove the claim by generalized induction on the variable n. Thus we assume (15) for $0 \leq n \leq N-1$, and show it for $n = N$. Using (14) with $n = N$,

$$\begin{aligned} 2^{2N+1} a_N &= (2N+2) \sum_{k=0}^{N} a_{N-k} a_k \\ &= 2(2N+2) a_0 a_N + (2N+2) \sum_{k=1}^{N-1} \frac{\lambda^{N-k}}{(2(N-k)+1)!} \frac{\lambda^k}{(2k+1)!}, \end{aligned}$$

and thus

$$a_N = \frac{\lambda^N}{(2N+1)!} \frac{1}{2^{2N+1} - 2(2N+2)} \sum_{k=1}^{N-1} \binom{2N+2}{2k+1},$$

and the result follows because $\sum_{k=0}^{N} \binom{2N+2}{2k+1} = 2^{2N+1}$. $\square$

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