Journal

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Goals

- Settle and prove the best bound for $H^1 \times H^1 \to L^2$. Almost had finished this.
- Settle whether or not L^∞ is really needed for $W_p[\phi\theta]$. We will multiply by H^1 and want to end in L^2.
- Learn how the finite element people implement H^1 norms in programs.
- Derive control $L^2 \to L^2_\nabla$ for integral kernels, as in Section 4.4 and A.2

Finite Element Method

We consider the partial differential equation

$$\frac{\partial}{\partial x}(p(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y}) + r(x,y)u(x,y) = f(x,y),$$

with $(x,y) \in D$, where D is a plane region with boundary $\mathcal{S}$. Boundary conditions of the form $u(x,y) = g(x,y)$ are imposed on a portion, $\mathcal{S}_1$, of the boundary. On the remainder of the boundary, $\mathcal{S}_2$, the solution $u(x,y)$ is required to satisfy

$$p(x,y)\frac{\partial u}{\partial x}(x,y)\cos\theta_1 + q(x,y)\frac{\partial u}{\partial y}(x,y)\cos\theta_2 + g_1(x,y)u(x,y) = g_2(x,y)$$

where θ_1 and θ_2 are the direction angles of the outward normal to the boundary at the point (x,y)

The finite element method approximates the solution by minimizing the functional I below over a smaller class of functions.

$$
\begin{aligned}
&I[w] \\
&= \int\int_D \{\frac{1}{2}[p(x,y)(\frac{\partial w}{\partial x})^2 + q(x,y)(\frac{\partial w}{\partial y})^2 - r(x,y)w^2] + f(x,y)w\}dxdy \\
&+ \int_{\mathcal{S}_2} \{-g_2(x,y)w + \frac{1}{2}g_1(x,y)w^2\}dS
\end{aligned}
$$

- The first step is to divide the region into a finite number of sections, or elements.
- The set of functions used for approximation is generally a set of piecewise polynomials of fixed degree in x and y. For example, $w(x, y) = a + bx + cy$
- For general discussion, suppose that the region has been subdivided into triangular elements. The collection of triangles is denoted D, and the vertices of these triangles are called nodes. The method seeks an approximation of the form

$$w(x, y) = \sum_{i=1}^{m} \gamma_i w_i(x, y)$$

where $w_i, w_2, \cdots, w_m$ are linearly independent piecewise-linear polynomials, and $\gamma_1, \gamma_2 \cdots, \gamma_m$ are constants.

Assume that we are approximating by means of the finite element method a boundary value problem whose solution u is sufficiently smooth, so that the V_h-interpolate $\Pi_h u$ is well-defined. Then, in view of the fundamental error bound, we have

$$||u - u_h|| \leq \inf_{v_h \in V_h} ||u - v_h|| \leq C||u - \Pi_h u||,$$

where u_h denotes the solution of the discrete problem and C is a constant independent of the subspace V_h.

Taking into account that we are essentially working with Sobolev norms $||\cdot||_{m,\Omega}$ with $m = 1$ or 2, and that $\Pi_h u|_K = \Pi_K u$ for all $K \in \tau_h$, we can write

$$||u - \Pi_h u||_{m,\Omega} = (\sum_{K \in \tau_h} ||u - \Pi_k u||^2_{m,K})^{\frac{1}{2}}$$

Therefore, the problem of finding an upper bound for $||u - u_h||$ is reduced to the "local" problem of evaluating quantities such as $||u - \Pi_k u||_{m,K}$.

According to the theoretical analysis, there exists a constant C such that, for all finite elements in the family, and all functions $u \in W^{k+1,p}(K)$,

$$||u - \Pi_K u||_{m,q,K} \leq C(h_K^n)^{\frac{1}{q}-\frac{1}{p}} h_K^{k+1-m} ||u||_{k+1,p,K}.$$

Setting $k = m = 1,\ p = q = 2$, we obtain

$$||\phi - \Pi_h \phi||_{1,K} \leq Ch||\phi||_{2,K}$$

and therefore

$$\inf_{\phi_h \in V_h} ||\phi - \phi_h||_{1,\Omega} \leq Ch||\phi||_{2,\Omega}$$

In addition, following the setting of the finite element method, we have

$$||u - u_h||_{0,\Omega} \leq Ch||u - u_h||_{1,\Omega}$$

So

$$||u - u_h||_{0,\Omega} \leq Ch^2||u||_{2,\Omega},$$

where C is independent of h.

In conclusion, we have the follow error estimation in finite element approximation by linear splines for the boundary value problem.

$$||u_h - u||_{H^1} \leq Ch||u''||_{L^2}$$

with some positive constant C.

This week, I am working on the report to summarize the main ideas and conclusions of finite element method, especially the error analysis. We hope that the idea of finite element method can help us in finding proper norms and accurate bounds for Prof. Martin's paper.

Derive the $L^2 \to L^2_\nabla$ norm bound on integral operator κ

This is a counterpart of what Prof. Martin has done in his paper. Yanran Chen and I are trying to apply the similar method on the $L^2 \to L^2_\nabla$ norm bound on integral operator κ. And the previous edition is the $L^2 \to L^2$ norm bound on integral operator κ

When we are working on the the $L^2 \to L^2_\nabla$ norm bound on integral operator κ and the $L^2 \to L^2_\nabla$ norm bound on error operator, we need to justify the use of the Leibniz integral rule on unbounded space. Since we are not discussing the functions, which are defined in a compact set. We need to place proper conditions on the functions. First, I read the proof of the Leibniz integral rule and then I ask Prof. Aizicovici for help.

If we have an integral of the form:

$$\int_{y_0}^{y_1} f(x,y)dy,$$

then for $x \in (x_0, x_1)$ the derivative of this integral is thus expressible:

$$\frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy = \int_{y_0}^{y_1} \frac{\partial}{\partial x} f(x,y)dy.$$

Provided that f and $\partial f/\partial x$ are both continuous over a region in the form $[x_0, x_1] \times [y_0, y_1]$.
This mean the integral is over a compact set.

If we need to get rid of this, we need to put an additional condition on f, which requires

$$\int\int \frac{\partial}{\partial x} f(x,y)dy$$

is well defined over the unbounded domain. However, this condition seems too strong for Prof. Martin's project. So, we need to find another condition which is not too strong to apply for unbounded case.

I am working on the final report, final presentation, and the journal.