Visualizing Low-Rank Tensors With Software

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Tensors: A tensor is a set of values arranged in a d-dimensional array, much like a matrix. In fact, a matrix is an example of a 2-dimensional tensor. If the tensor has the same number of rows, or entries, in every dimension, then we call that number the resolution, m. Thus, an $n \times n$ matrix has resolution-n. The following is an example of a tensor with dimension-3, resolution-2.

$$\begin{bmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{bmatrix}, \begin{bmatrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{bmatrix}$$

The Tensor Product: The tensor product, denoted $\otimes$, is analogous to the outer product of vectors. Thus, a tensor like the one above may have been achieved by:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1b_1c_1 & a_1b_2c_1 \\ a_2b_1c_1 & a_2b_2c_1 \end{bmatrix}, \begin{bmatrix} a_1b_1c_2 & a_1b_2c_2 \\ a_2b_1c_2 & a_2b_2c_2 \end{bmatrix}$$

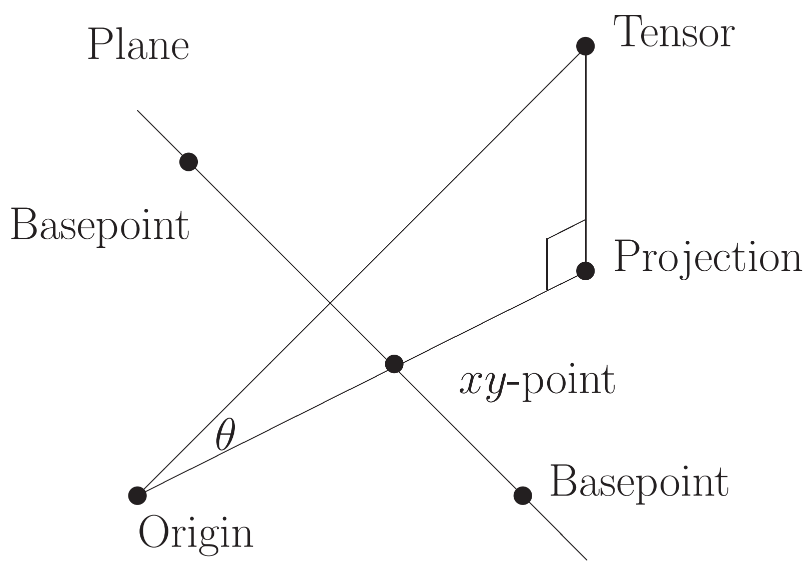
The Curse of Dimensionality: A tensor with d dimensions and resolution-m takes a computer m^d operations just to access. When m and d become large, computational cost becomes astronomical. This is called the Curse of Dimensionality.

The Separated Representation: The separated representation of a tensor is a potential answer to this problem. Here, a d-dimensional tensor with resolution-m is represented with error ϵ by a sum of r tensor-products of d one-dimensional vectors V with m entries in each vector. So, if s_l is a scalar, the separated representation of a tensor is given by

$$\sum_{l=1}^{r} s_l V_1^l \otimes V_2^l \otimes \ldots \otimes V_d^l$$

Now, a computer may carry out operations with a tensor with only rdm values to keep track of. However, this structure is not yet well understood.

To plot tensors on a 2-dimensional plane, we pick 3 basepoints that define the plane. Then a projection of the tensor onto the span of the basepoints is found and scaled into an xy-point on the plane. This xy-point represents the tensor in our 2-d image.



The angle θ is used to color-code the points plotted. A smaller θ means the tensor was closer to the plane, and a darker color is plotted.

In order to implement our plotting scheme and explore these separated representations quickly and easily, we wrote a program. It was written in the Python language, and included the following features:

- Plotting of colored points into a PostScript document.
- Option to plot a number of random points.
- Option to plot a random walk of points.
- Option to plot homotopies.
- Option to plot an alternating least squares fitting on a rectangular region.

Designing a GUI

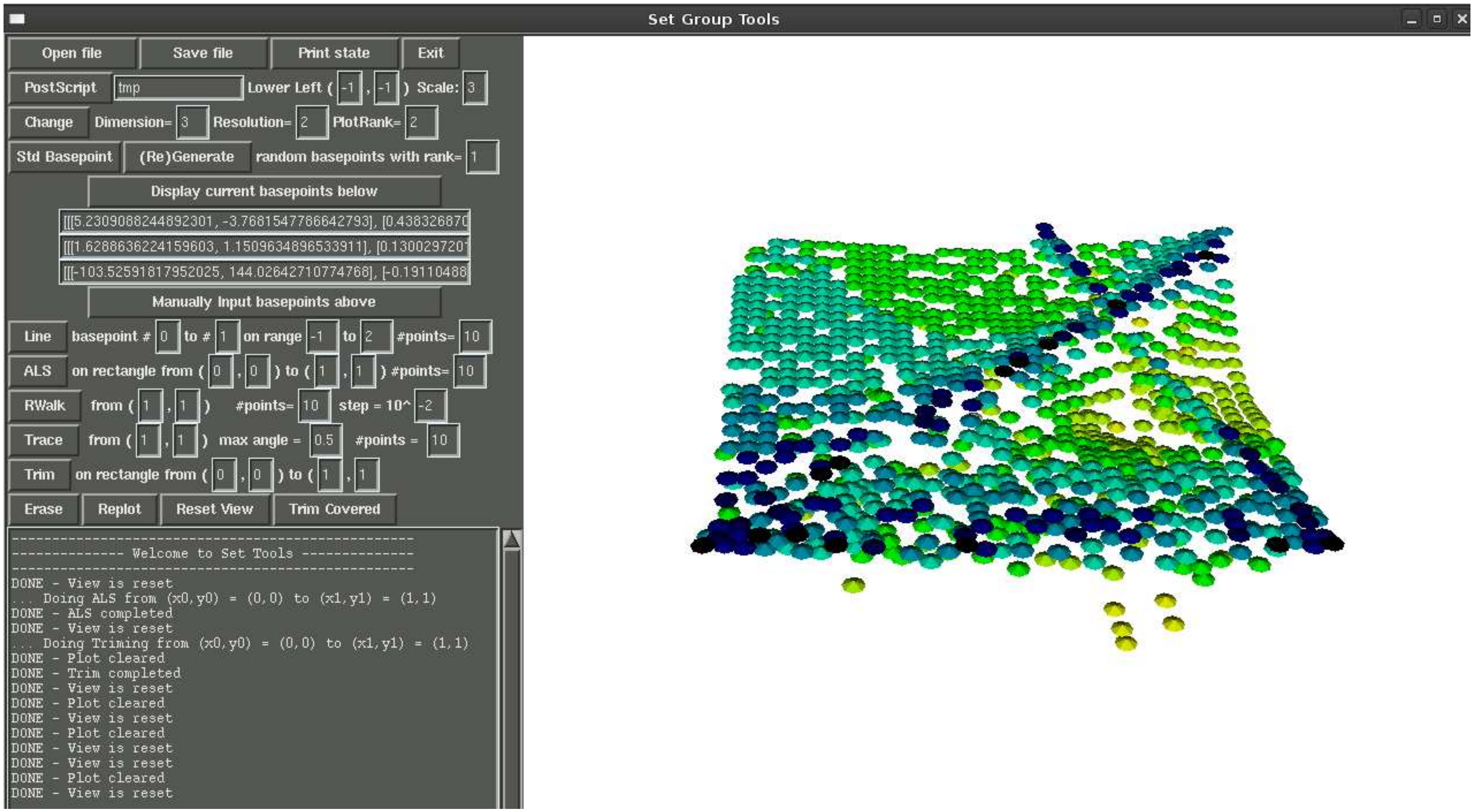
The program was functional, but unwieldy. To make it easier to run, and user-friendly, we designed a GUI to go with it. This was the main focus of our work so far this year. Some details considered were:

- A viewing window that shows up-to-date plots.
- The ability to zoom and pan plots.
- The ability to change any and all pertinent parameters on the fly.
- The ability to save and load previous work sessions.

The latest version includes many of the desired features, and more. It has:

- A GUI built with Tkinter, and featuring items one would expect from any software, such as clickable buttons.
- A viewing window built with the Visualization Tool Kit, that allows for the zooming, panning, and rotating of plots.
- A project save and load feature.
- A printable image generator.
- The ability to easily change any parameter, such as rank, dimension, etc.
- The easy choosing of basepoints, either randomly or by direct input.
- Customizable and easy-to-use functions, such as linepoints, alternating least squares, and random walk.
- Plotting and viewer options to trim the image or reset the camera.
- A status box that tracks and displays recent user activity.

Overview of Program



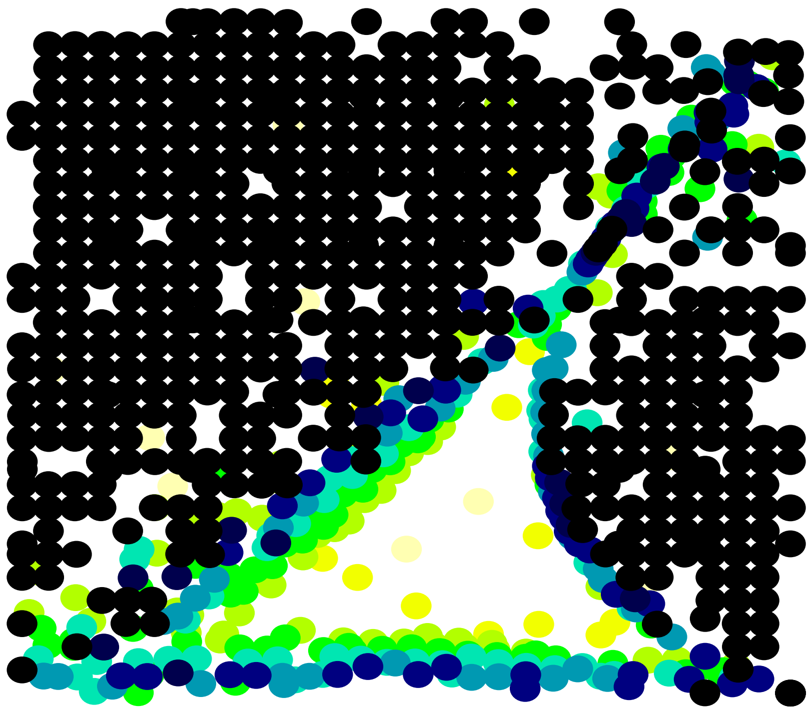
Linepoints: This function plots points in the viewing plane along one of the three lines defined by the basepoints. It makes it easier to see the basepoints in convoluted plots.

Alternating Least Squares: This function uses the alternating least squares method to find the closest tensors that can be represented in a given rectangular region of the viewing plane. This is the function we implement most often.

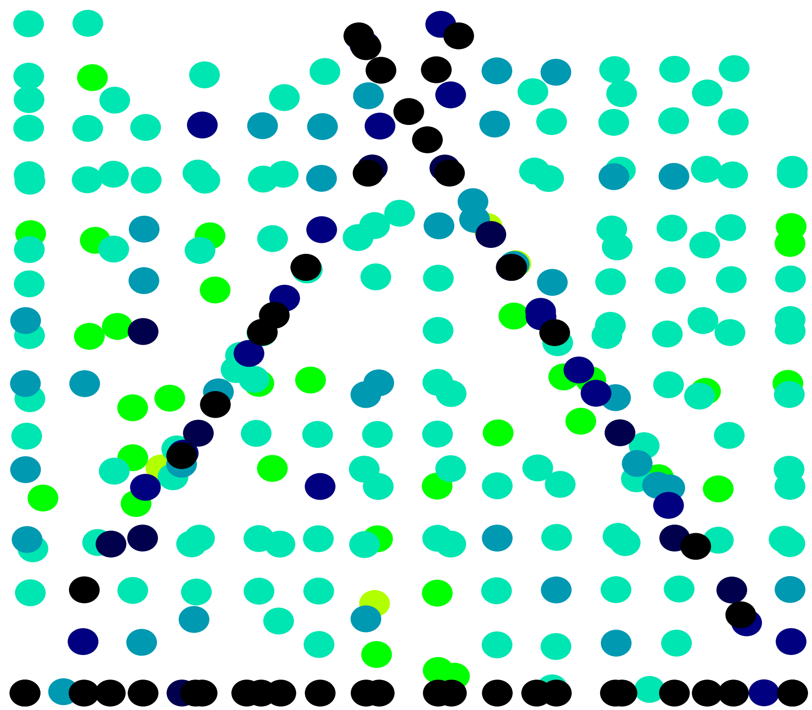
Random Walk: This function approximates and plots a random walk of tensors. Each new random tensor is within a given radius of the previous tensor. The new tensor is then plotted.

A Picture of G_3

G_3 is a representative tensor of a class of $3 \times 3 \times 3$ tensors in which each of them is truly rank 3 and cannot be approximated by rank-2 tensors.
G_3 is a combination of the three basepoints B_1, B_2, B_3 where
$B_1 = (e_1 + e_2) \otimes e_2 \otimes e_2, \quad B_2 = (e_1 - e_2) \otimes e_1 \otimes e_1, \quad B_3 = e_2 \otimes (e_1 + e_2) \otimes (e_1 - e_2)$
, i.e $G_3 = B_1 + B_2 + B_3$
By using the SetTools program, we can see how the picture of G_3 looks like:



D_3 is a representative tensor of a class of $3 \times 3 \times 3$ tensors in which each of them is rank 3 and can be approximated by rank-2 tensors.
D_3 is a combination of the three basepoints I_1, I_2, I_3 where
$I_1 = e_1 \otimes e_1 \otimes e_1, \quad I_2 = e_1 \otimes e_2 \otimes e_2, \quad I_3 = e_2 \otimes e_1 \otimes e_2$
, i.e $D_3 = I_1 + I_2 + I_3$
By using the SetTools program, we can see how the picture of D_3 looks like:



Accomplishments and Future Work

- We successfully made a tool that allows us to visualize the images of low rank tensor approximation.

- We had good pictures of $3 \times 3 \times 3$ tensors that support the theorem proposed by Vin De Silva and Lek-heng Lim.

- We are studying the pictures of tensors of higher rank to make some conjectures.

- We hope to draw some solid conclusions concerning the separated representation, now that the program is easy to use.

- We hope to add a help feature or user's guide to the program.