Journal of Nam Nguyen for Fall quarter 2007

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1 Week of September 4th,2007

1.1 LaTex

I tried to write something with LaTex and had some difficulties with its commands and typeset since I'm not very familiar with and good at LaTex. After many times trying, I could successfully compose my Mathematics Autobiography. I learned that LaTex is a very good word processor for writing and composing mathematical documents. I will work more with my LaTex advisor and I believe that my skills in LaTex will soon be improved.

1.2 Python program

From the website of Dr. Martins, I downloaded and read Chapter 1 and Chapter 2 (section 2.7) of Python book. I also downloaded, installed Python and ran a demo program in Python. Now I'm reading how to create a Python function and trying to program simple something with Python (solving a quaradic equation, for instance).

1.3 The Schrodinger Equation

I started reading the Schrodinger Equation but there were many points such as notations and some rules in Physics that I haven't understood yet. They would be very good topics for my further researches.

2 Week of September 25th,2007

This week, I read the paper "Tensor rank and the ill-posedness of the best low-rank approximation problem"(VIN DE SILVA and LEK-HENG LIM) and pointed out some definitions which were new to me:

- Tensor
- Rank of tensor
- Hyperdeterminant
- E-Y theorem

- Singular Value Decomposition

I also tried to give some examples for each definition, but I still not completely understand the tensors with order higher than 3. For the presentation week, I would talk about those things the may be the resuls #1 and #2.

3 Week of Octorber 2th,2007

Duy, Fan and I discussed about the Singular Value Decomposition(SVD), the E-Y theorem and their roles in the Tensor paper. We tried to give some specific examples for each of them as follow:

Singular Value Decomposition

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.236 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.447 & 0 & 0 & 0 & 0.894 \\ 0 & 0 & 0 & 1 & 0 \\ -0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix}$$

Then A = U.Σ.V^*. Moreover, both U and V are orthogonal matrices.

$$U.U^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V^*.(V^*)^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.447 & 0 & 0 & 0 & 0.894 \\ 0 & 0 & 0 & 1 & 0 \\ -0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix} \begin{pmatrix} 0 & 0.447 & 0 & 0 & -0.894 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.894 & 0 & 0 & 0 & 0.447 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We also discussed about the first two results. They are the most important ones of the tensor paper. In next week, we will present what we discussed this week.

4 Week of October 15th,2007

4.1 Tensor

An order-k tensor A $\in R^{d_1 \times d_2 \times \ldots \times d_k}$ can be simply considered as a vector in a k-dimentional space.
Example:

k = 0: A $\in R$ - a scalar
k = 1: A $\in R^{d_1}$ - a vector. A $= [x_1, x_2, ..., x_{d_1}]$ with index A_i
k = 2: A $\in R^{d_1 \times d_2}$ - a matrix.

$$\text{A} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ d_1 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & ... & d_2 \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & ... & x_{1d_2} \\ x_{21} & x_{22} & ... & x_{2d_2} \\ ... & ... & ... & ... \\ x_{d_1 1} & x_{d_1 2} & ... & x_{d_1 d_2} \end{pmatrix} \end{array} \text{with index} A_{ij}$$

k = 3: A $\in R^{d_1 \times d_2 \times d_3}$ - A is a cubic with index A_{ijk}

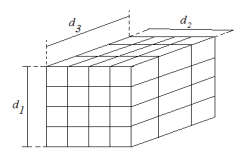

Fig 1: An 3-order tensor

4.2 Tensor product

From page 8, we get the formal definition of tensor product. Since $R^{d_1} \otimes R^{d_2} \otimes ... \otimes R^{d_k} \cong R^{d_1 \times d_2 \times \ldots \times d_k}$, the tensor product can be understood as:

$$x_1 \otimes x_2 \otimes ... \otimes x_k = [x^{(1)}_{j_1} ... x^{(k)}_{j_k}]^{d_1,\ldots,d_k}_{j_1,\ldots,j_k=1}$$

For example: k = 3, $d_1 = 4, d_2 = d_3 = 3$.

$$\overset{x^{(1)}}{\begin{pmatrix} x^{(1)}_1 \\ x^{(1)}_2 \\ x^{(1)}_3 \\ x^{(1)}_4 \end{pmatrix}} \qquad \overset{x^{(2)}}{\begin{pmatrix} x^{(2)}_1 \\ x^{(2)}_2 \\ x^{(2)}_3 \end{pmatrix}} \qquad \overset{x^{(3)}}{\begin{pmatrix} x^{(3)}_1 \\ x^{(3)}_2 \\ x^{(3)}_3 \end{pmatrix}}$$

then

$$x_1 \otimes x_2 \otimes ... \otimes x_k = [x^{(1)}_1 x^{(2)}_1 x^{(3)}_1, x^{(1)}_1 x^{(2)}_1 x^{(3)}_2, x^{(1)}_1 x^{(2)}_1 x^{(3)}_3, x^{(1)}_1 x^{(2)}_2 x^{(3)}_1, ..., x^{(1)}_4 x^{(2)}_3 x^{(3)}_3]$$

A tensor A $\in R^{d_1 \times d_2 \times ... \times d_k}$ is said to be decomposable if it can be written in the form

$$A = x_1 \otimes x_2 \otimes ... \otimes x_k$$

where $x_i \in R^{d_i}$ for all i = 1,..., k.
With matrcies

$$(L_1, ..., L_k).x_1 \otimes x_2 \otimes ... \otimes x_k = L_1 x_1 \otimes L_2 x_2 \otimes ... \otimes L_2 x_k$$

4.3 Some results of the paper

Best rank-r approximation to a tensor $A \in R^{d_1 \times d_2 \times ... \times d_k}$

$$||A - x_1 \otimes y_1 \otimes ... \otimes z_1 - ... - x_r \otimes y_r \otimes ... \otimes z_r||$$

or $argmin_{rank_{\otimes}(B) \leq r} ||A - B||$ (APPROX(A,r))

Result 1
APPROX(A,r) is **ill-posed for many r**. We will show that, regardless of the choice of norm, the problem of determining a best rank-r approximation for an order-k tensor in $R^{d_1 \times d_2 \times ... \times d_k}$ has no solution in general for $r = 2, ..min\{d_1, .., d_k\}$ and $k \geq 3$. In other words, the best low rank approximation problem for tensors is ill-posed for all orders (higher than 2), all norms, and many ranks.

Result 2
APPROX(A,r) is **ill-posed for many A**. We will show that the set of tensors that fail to have a best low rank approximation has positive volume. In other words, such failures are not rare if one randomly picks a tensor A in a suitable tensor space, then there is a non-zero probability that A will fail to have a best rank-r approximation for some $r < rank(A)$.

Result 3
Weak solutions to approx(A,r). We will propose a natural way to overcome the ill-posedness of the best rank-r approximation problem with the introduction of weak solutions and we characterize all weak solutions in the case $r = 2, k = 3$.

Result 4
Semialgebraic description of tensor rank. We will show that for any $d_1, .., d_k$, there exists a finite number of polynomial functions $\Delta_1, .., \Delta_m$, defined on $R^{d_1 \times d_2 \times ... \times d_k}$ that the rank of any A $\in R^{d_1 \times d_2 \times ... \times d_k}$ is completely determined by the signs of $\Delta_1(A), .., \Delta_m(A)$. We work out in the special case $R^2 \times 2 \times 2$

Result 5
Reduction. We will give techniques for reducing certain questions about tensors (orbits, invariant, limits) from high-dimensional tensor spaces to lower-

dimensional tensor spaces. For instance, if two tensors in $R^{c_1 \times c_2 \times \ldots \times c_k}$ lie in distinct $GL_{c_1,c_2,\ldots,c_k}(R)$-orbits, then they lie in distinct $GL_{d_1,d_2,\ldots,d_k}(R)$-orbits in $R^{d_1 \times d_2 \times \ldots \times d_k}$ for any $d_i \geq c_i$.

5 Week of October 22th,2007

Last week I read Theorem 1.1, 1.2 and 1.3. I also tried section 2.1 and section 2.2 but there was some things that I was not completely understand yet

1.$GL_{d_1,\ldots,d_k}$

2.$O_{d_1,\ldots,d_k}$

In this week, I will try to complete section 2.2 and continue reading section 2.3 and programming some simple things with Python.