# Journal of Duc Duy Nguyen

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## 1 Week 1, September 3 to 10

I read some things issued by Dr. Martin to get clear understanding about rules of research group. Although I have known how to use LaTeX before I came to Ohio University but the format are quietly new and different so I began to read some books about LaTeX, learn commands of LaTeX, I started writing my Mathematical Autobiography using LaTeX, and I had Dr. Martin correct if for me. Eventually, I have been getting used to writing some things using LaTeX.

## 2 Week 2, September 10 to 18

Dr. Martin gave to us the tensor paper *"Tensor Rank And The Ill-posedness Of The Best Low-Rank Approximation Problem"* of Vin De Silva and Lek-Heng Lim. Dr. Martin told us to read the paper in advanced and we would have a presentation about the tensor paper in next three weeks. He just required us to state the results of the paper but after spending time to look at the paper, we began to be worried because it was really difficult, there was a huge of new concepts, definitions we have never seen or known before, for example, an order-k tensor, low rank approximation, singular value decomposition (SVD), many symbols like $argmin_{rank_{\otimes}(B)\leq}||A-B||$ or

$$||A - x_1 \otimes y_1 \otimes \ldots \otimes z_1 - \ldots \otimes x_r \otimes y_r \otimes \ldots\ldots \otimes z_r||$$

## 3 Week 3, September 19 to 26

I began spending my time on the tensor paper. I used Google to search the paper of Eckart - Young about low rank approximation published in 1936 to get clear the original state of Eckart - Young theorem, but I could not find it. By Google I knew that to understand of Eckart - Young theorem I had to know the definition of SVD. Its (SVD) definition was stated as :
For a matrix $A \in M_{n\times p}(\mathbb{R})$ then there exist orthogonal matrices $U \in M_{n\times n}(\mathbb{R})$ and $V \in M_{p\times p}(\mathbb{R})$ such that $A = U\Sigma V$ where $\Sigma = diag(\sigma_1, \ldots \sigma_r, 0 \ldots, 0)$, where $r = rank(A)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$, and $\sigma_i$ are called *singular values* of A.
By using SVD decomposition of a matrix $A \in \mathbb{R}^{n\times m}$ we could find the

minimal for the following problem : $||r|| = \min ||Ax - b|| = \min ||U\Sigma Vx - b|| = \min ||\Sigma Vx - U^T b|| = \min ||z - c||$ where $z = \Sigma Vx$ and $c = U^T b$ and the last one is easier than the original problem.

# 4 Week 4, September 27 to October 2

Three of us gathered in Room 325, Morton Hall on Sunday and discussed many questions related to tensor paper. I presented for Nam, and Fan about the introduction of SVD theorem, tensor definition, Eckart - Young theorem. The Eckart - Young theorem states that if

$$A = U\Sigma V = \sum_{i=1}^{rank(A)} \sigma_i u_i \otimes v_i \text{ where } \sigma_i \geq \sigma_{i+1}$$

is the singular value decomposition of $A \in \mathbb{R}^{n\times m}$ then a best rank-r approximation of A is given by the first $r$ terms in the above sum. Meanwhile Nam and Fan were listening to me, gave their suggestions, critiqued many things about SDV theorem. They also raised so many questions that made all of us be really confused. Nam suggested me to give the clear definition about tensor, how it looked like in three dimension vector spaces. I gave them some kind of tensors, for example when we played with matrix vector space then tensors are matrices and tensor operation is matrix multiplication.

# 5 Week 5, October 3 to 10

We went to weekly seminar of research group, listened to the presentation of other members. Jyothsna Jakka gave a talk related to some function in physics, after that we gathered in Room 325 again and I talked about SVD, and discussed why people can discard some smallest singular values of a matrix $A \in \mathbb{R}^{n\times m}$ and Fan did give us a clear explanation. Nam began to present to us his part about the tensor definition, he gave us some more clear pictures of tensor in 1- dimension vector space, ..., 3-dimension vector spaces. At first, I intended to talk about SVD decomposition, and Eckart - Young theorem, and tensor definition but it was too long so three of us agreed that I would talk about SVD decomposition, Eckart - Young theorem would be of Fan's, Nam would talk about tensor definition, and some examples.

# 6 Week 6, October 11 to 18

This week, I started presenting for the group my part of the tensor paper about SVD. All the presentation were like I had talked with Nam and Fan. Moreover, I also talked about some application of SVD in data compression, using SVD to solve system of linear equation, finding inverse of matrix using SVD. Fan talked about Eckart - Young theorem, he gave the group an example about SVD and then he stated the Eckart - Young theorem.

# 7 Week 7 , October 19 to 26

I went to the seminar of of Dr.Todd Eisworth about "Down with Ramsey Theory"
I kept reading the tensor paper, trying to understanding some symbols in tensor paper. I prepared for the following problems :

$$rank_{\otimes}(A) := \min\{r|A = \sum_{i=1}^{r} u_i \otimes v_i \otimes \ldots \otimes z_i\}$$

I began to reading the theorem 1.1 which states that :

**Theorem 1.1** Let $d_1, d_2, d_3$ and $A_n \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ where $n \in \mathbb{N}$ be a sequences of tensors with $rank_{\otimes}(A_n) \leq 2$ and $\lim_{n\to\infty} A_n = A$, where the limit is taken in any topology. If the limiting tensor has rank higher than 2 then $rank_{\otimes}(A) = 3$, and there are pair of linear independent vector $x_i, y_i \in \mathbb{R}^{d_i}, \forall i = 1, 2, 3$ such that

$$A = x_1 \otimes x_2 \otimes y_3 + x_1 \otimes y_2 \otimes y_2 \otimes x_3 + y_1 \otimes x_2 \otimes x_3$$

Further more, the above result is not vacuous since :

$$A_n = n(x_1 + \frac{1}{n}y_1) \otimes n(x_2 + \frac{1}{n}y_2) \otimes n(x_3 + \frac{1}{n}y_3) - nx_1 \otimes x_2 \otimes x_3 \overset{n\to\infty}{\to} A \ (\star)$$

**Note** :
i) The boundary point of all order - 3 rank-2 tensor can be completely parameterized
ii) A sequence of order-3 rank-2 tensors can not "jump rank" by more than 1.
iii) The limiting tensor A in (*) is an example of a tensor that has no best rank -2 approximation.

# 8 Week 8, October 27 to November 2

I got ready for the main results of the tensor paper and presented the following results of the tensor paper :

**(6). Divergence of coefficients**. If $A_n \to A$ where rank $_\otimes(A) \geq r+1$ and $rank_\otimes(A_n) \leq r$ $\forall n \in \mathbb{N}$. If we write :

$$A_n = \lambda_{1,n}.u_{1,n} \otimes v_{1,n} \otimes w_{1,n} + \ldots + \lambda_{r,n} u_{r,n} \otimes v_{r,n} \otimes w_{r,n}$$

where $u_{i,n}, v_{i,n}, w_{i,n}$ are unit vectors. Then $\max\{|\lambda_{i,n}| : i = 1, ..., r\} \to +\infty$ Moreover at least two of coefficients sequences $\{\lambda_{i,n} : i = 1, ..., r\}$ are unbounded.

**(7) Maximum rank**. For $k \geq 3$ the maximum rank of an ordered-k tensor in $\mathbb{R}^{d_1 \times \ldots \times d_k}$ where $d_i \geq 2$ always exceeds $\min\{d_1, \ldots, d_k\}$ (but we have known that $rank_\otimes(M) \leq \min\{d_1, d_2\}$ where $M \in \mathbb{R}^{d_1 \times d_2}$)

**(8) Tensor rank can leap large gaps**. Conclusion ii) in the papragraph above does not generalize to rank $r > 2$ . We will show that a sequence of fixed rank tensor can converge to a limiting tensor of artribay higher rank.

**(9) Brëgman divergences do not help**.
For $F : \Delta \to R$ where $\Delta$ is a convex set. The Brëgman distance associated with $F$ for points $p, q \in \Delta$ is :

$$B_F(p||q) := F(p) - F(q) - \nabla F(q)(p - q)$$

The $B_F(p||q)$ does not satisfy the triangle inequality.
If we replace norm by any continuous measure of "nearest" (including non-metric Brëgman divergences), it does not change the ill-posedness of APPROX(A,r).

(10) We will construct a rich family of sequences of tensors with degenerate limit, labled by partial derivative operator. From that (*) is just a special case.

I also looked for some books about Python and started reading the Python language, went to the seminar "Learning a Function of Many Variables" of Dr. Martin.