Tensor Approximation Journal

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1 Standard Paper Examples

1.1 Paatero Example

From Paatero's work on degenerate PARAFAC models, consider equation (21):

$$\mathrm{X} = \left(\begin{matrix} 0 & 1 \\ 1 & d \end{matrix}\;\middle|\;\begin{matrix} e & 0 \\ 0 & h \end{matrix}\right). \tag{1}$$

If we want to consider the rank-2 case of X, we can separate X using Hackbusch's formula for a rank-2 $2 \times 2 \times 2$ tensor

$$X = \alpha\,(v_{1,1} \otimes v_{1,2} \otimes v_{1,3}) + \beta\,(v_{2,1} \otimes v_{2,2} \otimes v_{2,3}) \tag{2}$$

$$= \alpha \begin{pmatrix} 1 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ a_3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ b_3 \end{pmatrix} \tag{3}$$

$$= \frac{e}{W} \begin{pmatrix} 1 \\ \frac{de+W}{2e} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{de+W}{2e} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{-de+W}{2} \end{pmatrix} - \frac{e}{W} \begin{pmatrix} 1 \\ \frac{de-W}{2e} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{de-W}{2e} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{-de-W}{2} \end{pmatrix} \tag{4}$$

where $W = \sqrt{4eh + (de)^2}$.

It's good practice to verify these values are correct. To do so, multiply out each tensor product. This calculation results in the following:

$$X = \left(\begin{matrix} \alpha + \beta & \alpha a_2 + \beta + b_2 \\ \alpha a_1 + \beta b_1 & \alpha a_1 a_2 + \beta b_1 b_2 \end{matrix}\;\middle|\;\begin{matrix} \alpha a_3 + \beta b_3 & \alpha a_2 a_3 + \beta b_2 b_3 \\ \alpha a_1 a_3 + \beta b_1 b_3 & \alpha a_1 a_2 a_3 + \beta b_1 b_2 b_3 \end{matrix}\right). \tag{5}$$

After verifying the Hackbusch rank-2 form, the next step is to calculate S_X, the tensor representation resulting by transforming the vectors of X into polar coordinates using the transformations

$$x = \cos(\theta)\,, y = \sin(\theta)\,, \theta = \arctan(\frac{y}{x})\,.$$

This changes the tensor decomposition into vectors of the form $\mathbf{u}(\theta) = [\cos(\theta), \sin(\theta)]$ with their corresponding magnitudes, which allows us to express the above as

$$S_X = \frac{e}{W}\|v_{1,1}\|\|v_{1,2}\|\|v_{1,3}\|[\mathbf{u}(\theta_{1,1}) \otimes \mathbf{u}(\theta_{1,2}) \otimes \mathbf{u}(\theta_{1,3})] - \frac{e}{W}\|v_{2,1}\|\|v_{2,2}\|\|v_{2,3}\|[\mathbf{u}(\theta_{2,1}) \otimes \mathbf{u}(\theta_{2,2}) \otimes \mathbf{u}(\theta_{2,3})] \tag{6}$$

$$= z_0[\mathbf{u}(\theta_{1,1}) \otimes \mathbf{u}(\theta_{1,2}) \otimes \mathbf{u}(\theta_{1,3}) + z_1[\mathbf{u}(\theta_{2,1}) \otimes \mathbf{u}(\theta_{2,2}) \otimes \mathbf{u}(\theta_{2,3})]\,. \tag{7}$$

In order to get the tensor into standard form we need to apply Givens rotations to each corresponding vector of the tensor product. For our particular interest, Givens rotations can be expressed as matrix multiplication with the vector $[\cos(\theta), \sin(\theta)]$ for some θ. The Givens rotation for the vectors of the leading summand can be expressed as

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8}$$

where ϕ is the angle which rotates the vector $[\cos(\theta), \sin(\theta)]$ to $[1, 0]$.

In a general rank-r, dimension-d tensor setting, a series of Givens rotations for angles ϕ_j, denoted $G(\phi_j)$, can be applied to a tensor decomposition, $T = \sum_{i=1}^{r} z_i \bigotimes_{j=1}^{d} \mathbf{u}(\theta_{ij})$, in the following manner:

$$G(\phi_j)T = \sum_{i=1}^{r} z_i \bigotimes_{j=1}^{d} G(\phi_j)\mathbf{u}(\theta_{ij})\,. \tag{9}$$

We now adopt the above notation for angles of the tensor T to with i in the rank and j in dimension indexes and we denote a tensor T with a givens rotation, $G(\phi_j)$ for some ϕ_j, applied to it as GT.

As a more specific case, a symmetric Givens rotation can be applied to the tensor by fixing the argument ϕ. This would effectively be matrix multiplication by a single Givens rotation matrix, i.e. a single angle ϕ, to each vector in the tensor decomposition, giving us

$$G(\phi)T = \sum_{i=1}^{n} z_i \bigotimes_{j=1}^{d} G(\phi)\mathbf{u}(\theta_{i,j})\,. \tag{10}$$

In order to apply a Givens rotation about the first summand of our rank-2 example, we first find $\phi_{1,j}$, and apply a Givens rotation to $\mathbf{u}(\theta_{i,j})$ for each $i = 1, 2$ in the rank index to get a new vector, $[a_{i,j}, b_{i,j}]$, expressed as

$$\begin{pmatrix} \cos(\phi_{1,j}) & \sin(\phi_{1,j}) \\ -\sin(\phi_{1,j}) & \cos(\phi_{1,j}) \end{pmatrix} \begin{pmatrix} \cos(\theta_{i,j}) \\ \sin(\theta_{i,j}) \end{pmatrix} = \begin{pmatrix} a_{i,j} \\ b_{i,j} \end{pmatrix}\,. \tag{11}$$

Applying the corresponding Givens rotation to every $\theta_{i,j}$ will transform the vectors $v_{1,1} = v_{1,2} = v_{1,3} = [1, 0]$ and $v_{2,1}, v_{2,2}, v_{2,3}$ into

$$v_{2,j} = [\cos(\theta_{2,j} - \theta_{1,j}), \sin(\theta_{2,j} - \theta_{1,j})] = \mathbf{u}(\theta_{2,j} - \theta_{1,j})\,. \tag{12}$$

By taking the final solution values of $e = 30$, $d = .25$, and $h = -.23$ from section 6 of Paatero and applying Givens rotations, our new tensor decomposition, GS_X, is

$$GS_X = 39.230925 \bigotimes_{i=1}^{3} \mathbf{e}_1 - 11.140231(\mathbf{u}(.155898) \otimes \mathbf{u}(.155898) \otimes \mathbf{u}(.468673)) \tag{13}$$

$$\tilde{GS_X} = \bigotimes_{i=1}^{3} \mathbf{e}_1 - .283966(\mathbf{u}(.155898) \otimes \mathbf{u}(.155898) \otimes \mathbf{u}(.468673))\,. \tag{14}$$

Another calculation that is worth considering is the initial solution with a Givens rotation with respect to the final solution.

$$\begin{aligned} X =& 31.321873(\mathbf{u}(-0.052674) \otimes \mathbf{u}(-0.052674) \otimes \mathbf{u}(2.831479)) \\ &-4.967799(\mathbf{u}(-0.256033) \otimes \mathbf{u}(-0.256033) \otimes \mathbf{u}(1.616824))\,. \end{aligned} \tag{15}$$

1.2 Trigonometric Example

$$\sin\left(\sum_{j=1}^{d} x_j\right) = \sum_{j=1}^{d} \sin(x_j) \prod_{k=1, k\neq j}^{d} \frac{\sin(x_k + \alpha_k - \alpha_j)}{\sin(\alpha_k - \alpha_j)}\,, \tag{16}$$

which is valid for all choices of $\{\alpha_k\}$ such that $\sin(\alpha_k - \alpha_j) \neq 0$ for all $k \neq j$.

In the $d = r = 3$ case, by Mohlenkamp and Monzon, we know

$$\begin{aligned} \sin(x_1 + x_2 + x_3) =& \sin(x_1)\left(\frac{\sin(x_2 + \beta - \alpha)\sin(x_3 + \gamma - \alpha)}{\sin(\beta - \alpha)\sin(\gamma - \alpha)}\right) \\ &+ \left(\frac{\sin(x_1 + \alpha - \beta)}{\sin(\alpha - \beta)}\right)\sin(x_2)\left(\frac{\sin(x_3 + \gamma - \beta)}{\sin(\gamma - \beta)}\right) \\ &+ \left(\frac{\sin(x_1 + \alpha - \gamma)}{\sin(\alpha - \gamma)}\right)\left(\frac{\sin(x_2 + \beta - \gamma)}{\sin(\beta - \gamma)}\right)\sin(x_3) \end{aligned} \tag{17}$$

for all choices of α, β, γ satisfiying the above condition.

Changing to a $\mathbf{u}(\theta) = [\cos(\theta), \sin(\theta)]$ basis, the above can be expressed as

$$\begin{aligned} S = &\left(\frac{1}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}\right)[0,1]\otimes[\sin(\beta-\alpha),\cos(\beta-\alpha)]\otimes[\sin(\gamma-\alpha),\cos(\gamma-\alpha)] \\ &+\left(\frac{1}{\sin(\alpha-\beta)\sin(\gamma-\beta)}\right)[\sin(\alpha-\beta),\cos(\alpha-\beta)]\otimes[0,1]\otimes[\sin(\gamma-\beta),\cos(\gamma-\beta)] \\ &+\left(\frac{1}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}\right)[\sin(\alpha-\gamma),\cos(\alpha-\gamma)]\otimes[\sin(\beta-\gamma),\cos(\beta-\gamma)]\otimes[0,1] \end{aligned} \tag{18}$$

Using the standard basis $\mathbf{u}() = [1, 0]$ and the trigonometric identities $\sin(x) = \cos(\pi/2 - x)$ and $\cos(x) = \sin(\pi/2 - x)$. This simplifies the above to

$$\begin{aligned} S = &\left(\frac{1}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}\right)\mathbf{u}(\frac{\pi}{2})\otimes\mathbf{u}(\frac{\pi}{2}+\alpha-\beta)\otimes\mathbf{u}(\frac{\pi}{2}+\alpha-\gamma) \\ &+\left(\frac{1}{\sin(\alpha-\beta)\sin(\gamma-\beta)}\right)\mathbf{u}(\frac{\pi}{2}+\beta-\alpha)\otimes\mathbf{u}(\frac{\pi}{2})\otimes\mathbf{u}(\frac{\pi}{2}+\beta-\gamma) \\ &+\left(\frac{1}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}\right)\mathbf{u}(\frac{\pi}{2}+\gamma-\alpha)\otimes\mathbf{u}(\frac{\pi}{2}+\gamma-\beta)\otimes\mathbf{u}(\frac{\pi}{2})\,. \end{aligned} \tag{19}$$

We take the first term to be the leading term of the standard form (noting that a permutation can be made if this is not the case). Taking Givens rotations with respect to the angles of the directions of the first summand, we get the transformed tensor representation, GS, where

$$\begin{aligned} GS = &\left(\frac{1}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}\right)\bigotimes_{j=1}^{3}\mathbf{u}() \\ &+\left(\frac{1}{\sin(\alpha-\beta)\sin(\gamma-\beta)}\right)\bigotimes_{j=1}^{3}\mathbf{u}(\beta-\alpha) \\ &+\left(\frac{1}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}\right)\bigotimes_{j=1}^{3}\mathbf{u}(\gamma-\alpha)\,. \end{aligned} \tag{20}$$

Note that these Givens rotations are non-symmetric.

An interesting side to this example is to construct $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ so as to make $\sin(x) = \cos(\tilde{x})$. If we define

$$\begin{aligned} x &= \tilde{x} + \pi/2 \\ y &= \tilde{y} + \pi/2 + \alpha - \beta \\ z &= \tilde{z} + \pi/2 + \alpha - \gamma \end{aligned}$$

we then get the equality

$$\sin(x+y+z) = \sin\left(\tilde{x}+\tilde{y}+\tilde{z}+3\pi/2-\beta-\gamma+2\alpha\right)\,. \tag{21}$$

Alternatively, adding α to each angle would leave the tensor products as functions of α, β, γ. In order to do this, we need a symmetric Givens rotation with $\phi = -\alpha$ applied to each vector in the tensor product, resulting in

$$\begin{aligned} G_\alpha GS = &\left(\frac{1}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}\right)\bigotimes_{j=1}^{3}\mathbf{u}(\alpha) \\ &+\left(\frac{1}{\sin(\alpha-\beta)\sin(\gamma-\beta)}\right)\bigotimes_{j=1}^{3}\mathbf{u}(\beta) \\ &+\left(\frac{1}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}\right)\bigotimes_{j=1}^{3}\mathbf{u}(\gamma) \end{aligned} \tag{22}$$

This can simplify (21) by setting

$$x = \tilde{x}_\alpha + \pi/2 - \alpha$$
$$y = \tilde{y}_\alpha + \pi/2 - \beta$$
$$z = \tilde{z}_\alpha + \pi/2 - \gamma$$

to get

$$\sin(x+y+z) = \sin(\tilde{x}_\alpha + \tilde{y}_\alpha + \tilde{z}_\alpha + 3\pi/2 - \alpha - \beta - \gamma)\,. \tag{23}$$

1.2.1 The General Case

We now return to the generalized rank-d case (16). We can expand to get

$$\sin\left(\sum_{j=1}^{d} x_j\right) = \sum_{j=1}^{d} \sin(x_j) \prod_{k=1,k\neq j}^{d} \frac{\sin(x_k)\cos(\alpha_k - \alpha_j) - \sin(\alpha_k - \alpha_j)\cos(x_k)}{\sin(\alpha_k - \alpha_j)} \tag{24}$$

By change of basis to $[\cos(\theta), \sin(\theta)]$,

$$S = \sum_{j=1}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k - \alpha_j)}\right) \left(\bigotimes_{k=1,\neq j}^{d} \mathbf{u}(\pi/2 - (\alpha_k - \alpha_j))\right) \otimes_j [0,1]\,. \tag{25}$$

From the above, it follows that $\phi_{j,k} = \pi/2$ when $k = j$, and $\phi_{j,k} = \pi/2 - (\alpha_k - \alpha_j)$ when $k \neq j$. Again, we assume $j = 1$ to be the leading term of the standard form, noting that this can be altered up to permutations. Applying Givens rotations, denoted on the left-hand side by R, with respect to the angles in the $j = 1$ term for each $j \geq 1$,

$$RS = \left(\prod_{k=2}^{d} \frac{1}{\sin(\alpha_k - \alpha_1)}\right) \bigotimes_{j=1}^{d} \mathbf{e_1} + \sum_{j=2}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k - \alpha_j)}\right) \bigotimes_{j=1}^{d} \mathbf{u}(\alpha_j - \alpha_1)\,. \tag{26}$$

Letting $\beta_j = \alpha_j - \alpha_1$, we can write this as

$$RS = \left(\prod_{k=2}^{d} \frac{1}{\sin(\beta_k)}\right) \bigotimes_{j=1}^{d} \mathbf{e_1} + \sum_{j=2}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\beta_k - \beta_j)}\right) \bigotimes_{j=1}^{d} \mathbf{u}(\beta_j)\,, \tag{27}$$

with the condition that $\sin(\beta_1) \neq 0$ and $\sin(\beta_k - \beta_j) \neq 0$ for all $k \neq j$.

As in (22), we can apply a symmetric Givens rotation with α_1 as the argument, resulting in

$$R_\alpha RS = \sum_{j=1}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k - \alpha_j)}\right) \bigotimes_{j=1}^{d} \mathbf{u}(\alpha_j)\,, \tag{28}$$

with $\sin(\alpha_1) \neq 0$ and $\sin(\alpha_k - \alpha_j) \neq 0$ for all $k \neq j$.

1.2.2 Introducing Shifts in α_j

$$\sin\left(\sum_{j=1}^{d} x_j + \sum_{j=1}^{d} a_j\right) = \sum_{j=1}^{d} \sin(x_j + a_j) \prod_{k=1,k\neq j}^{d} \frac{\sin(x_k + a_k + \alpha_k - \alpha_j)}{\sin(\alpha_k - \alpha_j)} \tag{29}$$

$$= \sum_{j=1}^{d} \sin(x_j + a_j) \prod_{k=1,k\neq j}^{d} \frac{\sin(x_k + a_k)\cos(\alpha_k - \alpha_j) + \sin(\alpha_k - \alpha_j)\cos(x_k + a_k)}{\sin(\alpha_k - \alpha_j)} \tag{30}$$

Changing basis to $\mathbf{u}(x) = [\cos(x), \sin(x)]$, we can express, S_s, the tensor with shifts, as

$$S_s = \sum_{j=1}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k + \alpha_j)} \right) \bigotimes_{k=1,k\neq j}^{d} [\sin(a_k + \alpha_k - \alpha_j), \cos(a_k + \alpha_k - \alpha_j)] \otimes_j [\sin(a_j), \cos(a_j)] \quad (31)$$

Using the trigonometric identities

$$\sin(x - \pi/2) = \cos(x)$$
$$\cos(x - \pi/2) = \sin(x)$$

we can rewrite S_s as

$$S_s = \sum_{j=1}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k + \alpha_j)} \right) \bigotimes_{k=1,k\neq j}^{d} \mathbf{u}(\pi/2 - (a_k + \alpha_k - \alpha_j)) \otimes_j \mathbf{u}(\pi/2 - a_j) \quad (32)$$

Note the similarity to (25) in the general case. By applying a Givens rotation, G, with respect to the angles in the j=1 term for each j, our new tensor representation becomes

$$GS_s = \prod_{k=2}^{d} \frac{1}{\sin(\alpha_k - \alpha_1)} \bigotimes_{k=1}^{d} \mathbf{e_1} + \sum_{j=2}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k + \alpha_j)} \right) \bigotimes_{k=1}^{d} \mathbf{u}(a_k + \alpha_k - \alpha_1) \quad (33)$$

As in the general case, we can apply a symmetric Givens rotation to the tensor with respect to the angle $-(a_1 + \alpha_1)$, which will result in

$$G_\alpha GS_s = \sum_{j=1}^{d} \left(\prod_{k=1,k\neq j}^{d} \frac{1}{\sin(\alpha_k + \alpha_j)} \right) \bigotimes_{k=1,k\neq j}^{d} \mathbf{u}(a_k + \alpha_j) \otimes_j \mathbf{u}(a_j + \alpha_j - (a_k + \alpha_k)) \quad (34)$$

$$= \sum_{j=1}^{d} \bigotimes_{k=1,k\neq j}^{d} \frac{\mathbf{u}(a_k + \alpha_j)}{\sin(\alpha_k + \alpha_j)} \otimes_j \mathbf{u}(a_j + \alpha_j - (a_k + \alpha_k)) \quad (35)$$

1.3 Laplacian

Consider the separated, unitary tensor representation, T,

$$T = \left(\sum_{j=1}^{d} b_j^2 \right)^{-1/2} \left(\sum_{j=1}^{d} b_j \left(\bigotimes_{i=1,\neq j}^{d} \mathbf{e}_1 \right) \otimes_j \mathbf{e}_2 \right), \quad (36)$$

where b_j are arbitrary but not 0.

Assume an arbitrary $b_k = \max(b_j)$ for all j. Then, performing a Givens rotation by $-\pi/2$ in the k-th direction , we get the separated transformation

$$GT = \left(\sum_{j=1}^{d} b_j^2 \right)^{-1/2} \left(b_k \left(\bigotimes_{i=1}^{d} \mathbf{e}_1 \right) + \sum_{j=1,j\neq k}^{d} b_j \left(\bigotimes_{i=1,i\neq j,k}^{d} \mathbf{e}_1 \right) \otimes_j \mathbf{e}_2 \otimes_k (-\mathbf{e}_2) \right) \quad (37)$$

$$= \left(\sum_{j=1}^{d} b_j^2 \right)^{-1/2} \left(b_k \left(\bigotimes_{i=1}^{d} \mathbf{e}_1 \right) + \sum_{j=1,j\neq k}^{d} (-b_j) \left(\bigotimes_{i=1,i\neq j,k}^{d} \mathbf{e}_1 \right) \otimes_j \mathbf{e}_2 \otimes_k \mathbf{e}_2 \right). \quad (38)$$

1.4 Damping

The parallel factor decomposition (PARAFAC) is a common representation of a tensor using three matrices, A, B, and C, all of column length R and varying numbers of rows. A tensor, T, can be represented in PARAFAC notation by

$$T = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r \tag{39}$$

where $\mathbf{a}_r$ represents the r-th column vector of the matrix A, and similarly for $\mathbf{b}_r$ and $\mathbf{c}_r$.

In both `http://www.eurasip.org/Proceedings/Eusipco/Eusipco2005/defevent/papers/cr1102.pdf` and [?, NA-LA-KI:2008], a PARAFAC example is observed with

$$A = \begin{bmatrix} 1 & \cos(\theta) & 0 \\ 0 & \sin(\theta) & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & \sqrt{2}\cos(\theta) & 0 \\ 0 & \sin(\theta) & 1 \\ 0 & \sin(\theta) & 0 \end{bmatrix}, C = I_{3\times 3}\,. \tag{40}$$

We can express this tensor in the PARAFAC decomposition by

$$T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \otimes \begin{bmatrix} \sqrt{2}\cos(\theta) \\ \sin(\theta) \\ \sin(\theta) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{41}$$

In order to achieve the standard form, we need to perform a Givens rotation on the second direction in order to satisfy the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \sqrt{(2)}\cos(\theta) \\ \sin(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} \sqrt{(2)}\cos(\theta) \\ a \\ 0 \end{bmatrix}. \tag{42}$$

The choice of $\phi = \pi/4$ achieves the condition in (42). We then apply the Givens rotation matrix to the second direction of the third summand (as well as trivially to the second direction of the first summand). The resulting tensor representation is

$$T = 3\bigotimes_{i=1}^{3} \mathbf{e}_1 + \sqrt{2}\mathbf{u}(\theta) \otimes \mathbf{u}(\theta) \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{u}(\pi/2, -\pi/4) \otimes \mathbf{e}_3\,. \tag{43}$$

$\mathbf{u}(\pi/2, -\pi/4) = [0, 1/\sqrt{2}, -1/\sqrt{2}]$ $\langle \mathbf{u}(\pi/2, -\pi/4), \mathbf{u}(\theta)\rangle = \sin(\theta)/\sqrt{2}$ It's interesting to note that in [?, NA-LA-KI:2008], the authors make reference to the collinearity of the column vectors of $\mathbf{a}_r$ and $\mathbf{b}_r$ as $\theta \to 0$, however, in (43), if we take $\theta \to 0$, we get

$$T(0) = 3\bigotimes_{i=1}^{3} \mathbf{e}_1 + \sqrt{(2)}\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{u}(\pi/2, -\pi/4) \otimes \mathbf{e}_3\,, \tag{44}$$

which is a tensor with two collinear directions in two summands and an orthonormal summand in the other direction, and a completely orthonormal third summand.

1.4.1 Enhanced Line Search Example, 2008

In following with the example above, another example from [?][http://epubs.siam.org/doi/abs/10.1137/06065577] has similar structure but one added complexity. The example changes A and B to

$$A = \begin{bmatrix} 1 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & \sin(\theta) & 1 & \cos(\theta) \end{bmatrix}, B = \begin{bmatrix} 3 & \sqrt{2}\cos(\theta) & 0 & \sin(\theta) \\ 0 & \sin(\theta) & 1 & \cos(\theta) \\ 0 & \sin(\theta) & 0 & \sin(\theta) \end{bmatrix}, \tag{45}$$

and allows C and D to be randomly generated matrices. Note that as $\theta \to 0$, there are two sets of column vectors in both A and B that become collinear. The tensor can be represented as

$$T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \mathbf{c}_1 \otimes \mathbf{d}_1 + \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \otimes \begin{bmatrix} \sqrt{2}\cos(\theta) \\ \sin(\theta) \\ \sin(\theta) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{c}_2 \otimes \mathbf{d}_2$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{c}_3 \otimes \mathbf{d}_3 + \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix} \otimes \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix} \otimes \mathbf{c}_4 \otimes \mathbf{d}_4 \tag{46}$$

$$= 3\left(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{c}_1 \otimes \mathbf{d}_1\right) + \sqrt{2}\left(\mathbf{u}(\theta) \otimes \mathbf{u}(\theta) \otimes \mathbf{c}_2 \otimes \mathbf{d}_2\right) \tag{47}$$

$$+ \mathbf{e}_2 \otimes \mathbf{u}(\pi/2, -\pi/4) \otimes \mathbf{c}_3 \otimes \mathbf{d}_3 + \mathbf{u}(\pi/2 - \theta) \otimes \begin{bmatrix} \sin(\theta) \\ \cos(\theta - \pi/4) \\ \sin(\theta - \pi/4) \end{bmatrix} \otimes \mathbf{c}_4 \otimes \mathbf{d}_4 .$$

In this example, as $\theta \to 0$, the first two directions of the first two summands approach collinearity, as well as the first two directions of the third and fourth summands. It's been documented that when respective directions of different summands are close to each other, the approximation is more difficult to achieve numerically, or takes more iterations. The difference between this particular tensor and its predecessor is something to explore.

1.5 Blind Source Separation

The next example we examined is found in section 4 of `ftp://ftp.esat.kuleuven.be/pub/SISTA/delathauwer/reports/ldl-99-26.pdf` (LA-MO-VA2000) and section 6.3 of LI-NA-GL2015. The example starts with two source signals, $x_1(t), x_2(t)$, mixes them using a mixing matrix, M, and attempts to reconstruct the original signals. We are given $X(t)$ as

$$X(t) = \begin{cases} x_1(t) = \sqrt{2}\sin(t) \\ x_2(t) = \begin{cases} 1, \text{ if } t \in [k\pi, k\pi + \pi/2] \\ -1, \text{ if } t \in [k\pi + \pi/2, (k+1)\pi] \end{cases} \end{cases}, \tag{48}$$

over the interval $[0, 4\pi]$, and M, the mixing matrix, as

$$M = \frac{1}{4}\begin{bmatrix} -1 & -3\sqrt{3} \\ 3\sqrt{3} & -5 \end{bmatrix}, \tag{49}$$

such that $Z(t) = MX(t)$. The goal of the example is to find $V \in \mathbb{R}^{2\times 2}$ so that $VZ(t) = X(t)$. This relationship implies

$$V = M^{-1} = \frac{1}{8}\begin{bmatrix} -5 & 3\sqrt{3} \\ -3\sqrt{3} & -1 \end{bmatrix} \tag{50}$$

$$= SJS^{-1} = \begin{bmatrix} \frac{2+i\sqrt{23}}{3\sqrt{3}} & \frac{2-i\sqrt{23}}{3\sqrt{3}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-3-i\sqrt{23}}{8} & 0 \\ 0 & \frac{-3+i\sqrt{23}}{8} \end{bmatrix} \begin{bmatrix} \frac{-3i}{2}\sqrt{\frac{3}{23}} & \frac{1}{2} + \frac{i}{\sqrt{23}} \\ \frac{3i}{2}\sqrt{\frac{3}{23}} & \frac{1}{2} + \frac{i}{\sqrt{23}} \end{bmatrix}. \tag{51}$$

It's worth noting the eigenvalues of V are complex conjugates.

The rotation created by the mixing matrix can also be described using the fourth-order cumulant tensors $\mathcal{C}_Z, \mathcal{C}_X$ with respect to Z and X by

$$\mathcal{C}_Z = \sum_{r=1}^{2} (\mathcal{C}_X)_{rrrr} \bigotimes_{i=1}^{4} \mathbf{v}_r \tag{52}$$

where $\mathbf{v}_r$ refers to the r-th column vector of V. Due to LA-MO-VA2000, $\mathcal{C}_X$ is diagonal with $(\mathcal{C}_X)_{1111} = -3/2$ and $(\mathcal{C}_X)_{2222} = -2$. Therefore, we can compute (50) as

$$\mathcal{C}_Z = \left(\frac{-3}{2}\right)\left(\frac{\sqrt{13}}{4}\right)^4 \bigotimes_{i=1}^{4} \mathbf{u}(\theta_1) + (-2)\left(\frac{\sqrt{7}}{4}\right)^4 \bigotimes_{i=1}^{4} \mathbf{u}(\theta_2) \tag{53}$$

$$= \left(\frac{-507}{512}\right) \bigotimes_{i=1}^{4} \mathbf{u}(\theta_1) + \left(\frac{-49}{128}\right) \bigotimes_{i=1}^{4} \mathbf{u}(\theta_2)\,, \tag{54}$$

where $\theta_1 = .804634$ and $\theta_2 = -.190126$. Using a givens rotation, $G(\theta_2)$ on (54), the standard form can be expressed by

$$G(\theta_2)\mathcal{C}_Z = \frac{-49}{128} \bigotimes_{i=1}^{4} \mathbf{e_1} + \frac{-507}{512} \bigotimes_{i=1}^{4} \mathbf{u}(-.994759)\,. \tag{55}$$