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	15	1
	18	2
	17	3
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	15	6
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Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/3 1. (a) Find the orthogonal projection of $\vec{u} = \langle -1, 3\rangle$ onto $\vec{v} = \langle 5, 2\rangle$ (which is denoted $\text{proj}_{\vec{v}}\vec{u}$). [similar to 11.3 #24]

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{1}{29}\langle 5, 2\rangle = \langle \frac{5}{29}, \frac{2}{29}\rangle.$$

/3 (b) Write the vector and parametric equations for the line that passes through the point $P = (3, -5, 5)$ and is parallel to $\vec{d} = \langle 3, 1, 2\rangle$. [similar to 11.5 # 5] The vector equation is

$$\vec{\ell}(t) = \overrightarrow{0P} + t\vec{d} = \langle 3, -5, 5\rangle + t\langle 3, 1, 2\rangle,$$

so the parametric equations are

$$x = 3 + 3t$$
$$y = -5 + t \text{ and}$$
$$z = 5 + 2t\,.$$

/9 (c) Find the equation in general form for the plane that passes through the points $P = (1, -1, 1)$, $Q = (2, 2, 1)$, and $R = (2, 3, 2)$. [similar to 11.6 # 9] Select two vectors in the plane, such as

$$\vec{u} = \overrightarrow{PQ} = \langle 2, 2, 1\rangle - \langle 1, -1, 1\rangle = \langle 1, 3, 0\rangle$$
$$\vec{v} = \overrightarrow{RQ} = \langle 2, 2, 1\rangle - \langle 2, 3, 2\rangle = \langle 0, -1, -1\rangle.$$

Compute a normal vector

$$\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 0 \\ 0 & -1 & -1 \end{vmatrix} = (-3\vec{i} + 0\vec{j} - 1\vec{k}) - (0\vec{k} - 1\vec{j} + 0\vec{i}) = -3\vec{i} + 1\vec{j} - 1\vec{k} = \langle -3, 1, -1\rangle.$$

Using P as the base point, an equation in standard form is

$$0 = \vec{n}\cdot(\vec{x} - \overrightarrow{0P}) = -3(x-1) + 1(y+1) - 1(z-1)$$

so the equation in general form is

$$-3x + y - z = -5.$$

/3 2. (a) Find an equation for the tangent line to the graph of $\vec{r}(t) = \langle \sin(t), 3\cos(5t) \rangle$ at $t = \pi/4$. [similar to 12.2 #24] We compute $\vec{r}'(t) = \langle \cos(t), -15\sin(5t) \rangle$, so

$$\begin{aligned} \ell(t) = \vec{r}(\pi/4) + t\vec{r}'(\pi/4) &= \langle \sin(\pi/4), 3\cos(5\pi/4) \rangle + t\langle \cos(\pi/4), -15\sin(5\pi/4) \rangle \\ &= \langle 1/\sqrt{2}, -3/\sqrt{2} \rangle + t\langle 1/\sqrt{2}, 15/\sqrt{2} \rangle \end{aligned}$$

/3 (b) Find an equation for the normal line to the surface $2x^2 + y^2 + 3z^3 = 27$ at the point $P = (-1, 1, 2)$. [similar to 13.7 #21] Consider the surface as the level surface of a function $f(x, y, z)$ and compute the gradient

$$\nabla f(x, y, z) = \langle 4x, 2y, 9z^2 \rangle.$$

The gradient is orthogonal to the surface, so an equation for the normal line is

$$l(t) = \langle -1, 1, 2 \rangle + t\nabla f(-1, 1, 2) = \langle -1, 1, 2 \rangle + t\langle -4, 2, 36 \rangle.$$

/12 3. Find the critical points of the function $f(x, y) = \frac{1}{3}x^3 - x + \frac{1}{3}y^3 - 4y$. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, relative minimum, or saddle point. [similar to 13.8 #12] Compute the gradient

$$\nabla f(x, y) = \langle x^2 - 1, y^2 - 4 \rangle$$

and set equal to $\langle 0, 0 \rangle$. The first component gives $x = \pm 1$ and the second component gives $y = \pm 2$, so the critical points are $(-1, -2)$, $(-1, 2)$, $(1, -2)$, and $(1, 2)$.

$$f_{xx}(x, y) = 2x, \quad f_{yy}(x, y) = 2y, \quad \text{and } f_{xy}(x, y) = 0,$$

so $D(x, y) = 2x2y - 0 = 4xy$. Plugging in each point gives

$$\begin{aligned} D(-1, -2) = 8 > 0 \quad \text{and} \quad f_{xx}(-1, -2) = -2 < 0 \quad &\Rightarrow \quad \text{maximum} \\ D(-1, 2) = -8 < 0 \quad &\Rightarrow \quad \text{saddle} \\ D(1, -2) = -8 < 0 \quad &\Rightarrow \quad \text{saddle} \\ D(1, 2) = 8 > 0 \quad \text{and} \quad f_{xx}(1, 2) = 2 > 0 \quad &\Rightarrow \quad \text{minimum} \end{aligned}$$

/3 4. (a) Switch the order of integration in the iterated integral $\int_0^1 \int_{5-5x}^{5-5x^2} dydx$ to give another iterated integral that computes the same area. (You do not need to compute the integral.) [14.1 #18] As x ranges from 0 to 1, y goes from 5 to 4. Solving $y = 5 - 5x$ and $y = 5 - 5x^2$ for x yield $x = 1 - y/5$ and $x = \sqrt{1 - y/5}$, so we have

$$\int_0^5 \int_{1-y/5}^{\sqrt{1-y/5}} dxdy$$

/3 (b) Rewrite the integral $\int_0^2 \int_x^{\sqrt{8-x^2}} (x+y)dy\,dx$ in polar coordinates. (You do not need to compute the integral.) [similar to 14.3 #13] The region is a sector of a disc, with $\pi/4 \leq \theta \leq \pi/2$ and $0 \leq r \leq \sqrt{8}$. We thus get

$$\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} (r\cos(\theta) + r\sin(\theta))r dr\, d\theta$$

/11 5. Set up and evaluate the double integral to find the volume between $f_1(x,y) = \sin(x)\cos(y)$ and $f_2(x,y) = \cos(x)\sin(y) + 2$ over the triangle with corners $(0,0)$, $(0,\pi)$, and (π,π).

[similar to 14.6 #7]

$$\begin{aligned}
\int_0^\pi \int_x^\pi (f_2(x,y) - f_1(x,y))\, dydx &= \int_0^\pi \int_x^\pi (\cos(x)\sin(y) + 2 - \sin(x)\cos(y))\, dydx \\
&= \int_0^\pi -\cos(x)\cos(y) + 2y - \sin(x)\sin(y)\Big|_x^\pi\, dx \\
&= \int_0^\pi (-\cos(x)\cos(\pi) + 2\pi - \sin(x)\sin(\pi)) - (-\cos(x)\cos(x) + 2x - \sin(x)\sin(x))\, dx \\
&= \int_0^\pi 2\pi + 1 - 2x + \cos(x)dx = (\pi+1)x - x^2 + \sin(x)\Big|_0^\pi \\
&= (2\pi+1)\pi - \pi^2 + 0 - 0 = \pi^2 + \pi\,.
\end{aligned}$$

6. Consider the vector field $\vec{F} = \langle 2xyz, x^2z, x^2y\rangle$.
   [similar to 15.3 #19]

/3 (a) Find a potential function f for $\vec{F}$.
We know $\vec{F} = \langle f_x, f_y, f_z\rangle$, so antidifferentiating each partial derivative yields $f(x, y, z) = x^2yz + C_1(y, z) = x^2yz + C_2(x, z) + x^2yz = C_3(x, y)$ and we can take $C_1(y, z) = C_2(x, z) = C_3(x, y) = 0$, so $f(x, y, z) = x^2yz$ is a potential function.

/3 (b) Use the Fundamental Theorem of Line Integrals to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is a curve going from $(1, 2, 3)$ to $(-1, 5, -2)$.
By the theorem,

$$\int_C \vec{F} \cdot d\vec{r} = f(-1, 5, -2) - f(1, 2, 3) = (-1)^2(5)(-2) - 1^2(2)(3) = -10 - 6 = -16.$$

7. Let $\mathcal{S}$ be the plane $z = x + 3y$ over the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$.
   [similar to 15.5 #18]

/3 (a) Find a parameterization $\vec{r}(u, v)$ for $\mathcal{S}$.
We can let $\vec{r}(u, v) = \langle u, v(1 - u), u + 3v(1 - u)\rangle$ with $u \in [0, 1]$ and $v \in [0, 1]$.

/7 (b) Set up the integral to compute the surface area of $\mathcal{S}$. (You do not need to evaluate the integral.)
We have

$$\vec{r}_u = \langle 1, -v, 1 - 3v\rangle$$
$$\vec{r}_v = \langle 0, 1 - u, 3(1 - u)\rangle$$

so

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -v & 1-3v \\ 0 & 1-u & 3(1-u) \end{vmatrix} = \langle -3v(1 - u) - (1 - 3v)(1 - u), -3(1 - u), 1 - u\rangle$$
$$= \langle -(1 - u), -3(1 - u), 1 - u\rangle$$
$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{(1 - u)^2 + 9(1 - u)^2 + (1 - u)^2} = \sqrt{11}|1 - u|$$

Thus the surface area integral is

$$S = \iint_{\mathcal{S}} dS = \iint_R \|\vec{r}_u \times \vec{r}_v\| dA = \int_0^1 \int_0^1 \sqrt{11}|1 - u| du dv$$

Note: The solution above follows the method in section 15.5, from which this problem was taken, and where the domain of (u, v) is implicitly assumed to be a rectangle. Many students used the simpler parameterization $\vec{r}(u, v) = \langle u, v, u + 3v\rangle$ with $u \in [0, 1]$ and $v \in [0, 1 - u]$, which then leads to the surface area integral $\int_0^1 \int_0^{1-u} \sqrt{11} dv du$.

8. Let $\vec{F} = \langle -y, x\rangle$, C be the unit circle, $\vec{r}(t)$ be a counterclockwise parameterization of C, and R be the unit disc.

/4 (a) Compute $\oint_C \vec{F}\cdot d\vec{r}$

Parameterize the circle by $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$ for $t \in [0, 2\pi]$ so $\vec{r}'(t) = \langle -\sin(t), \cos(t)\rangle$ and $\vec{F} = \langle -\sin(t), \cos(t)\rangle$. We then have

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = 2\pi$$

/4 (b) Compute $\oint_C \vec{F}\cdot \vec{n}\, ds$

We can convert $\vec{n} ds = \langle\cos(t), \sin(t)\rangle dt$, so

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle\cos(t), \sin(t)\rangle dt$$
$$= \int_0^{2\pi} -\sin(t)\cos(t) + \sin(t)\cos(t) dt = \int_0^{2\pi} 0 dt = 0$$

/4 (c) Compute $\iint_R \text{curl}\vec{F}\, dA$

$\text{curl}\vec{F} = 1 - (-1) = 2$. Since we know that the area of the unit disc is π, we have

$$\iint_R \text{curl}\vec{F}\, dA = 2\iint_R dA = 2\pi$$

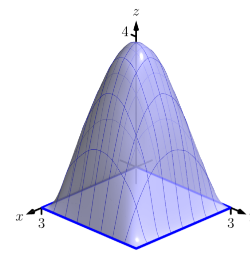
/3 (d) Compute $\iint_R \text{div}\vec{F}\, dA$

$\text{div}\vec{F} = 0 + 0 = 0$, so $\iint_R \text{div}\vec{F}\, dA = 0$.

/4 (e) Explain how these integrals relate to Green's theorem, Stokes' theorem, and/or the Divergence theorem.

By Green's theorem, $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\vec{F}\, dA$ and by the (2-dimensional) Divergence theorem, $\oint_C \vec{F}\cdot \vec{n}\, ds = \iint_R \text{div}\vec{F}\, dA$.

9. Let D be the domain enclosed by $z = xy(3-x)(3-y)$ and $z = 0$,
let $\mathcal{S}$ be the surface of this domain, and
let $\vec{F} = \langle 5x, 7y, 11+z \rangle$.



/10 (a) Set up the integral $\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$. (You do not need to evaluate the integral.)
[similar to 15.7 #7]
We can parameterize the top surface $\mathcal{S}_{\text{top}}$ by
$\vec{r}(u,v) = \langle u, v, uv(3-u)(3-v) \rangle$ for $u \in [0,3]$ and $v \in [0,3]$, so

$$\begin{aligned}
\vec{r}_u &= \langle 1, 0, (3-2u)v(3-v) \rangle \\
\vec{r}_v &= \langle 0, 1, (3-2v)u(3-u) \rangle \\
\vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & (3-2u)v(3-v) \\ 0 & 1 & (3-2v)u(3-u) \end{vmatrix} = \langle -(3-2u)v(3-v), -(3-2v)u(3-u), 1 \rangle \\
\vec{F} &= \langle 5u, 7v, 11 + uv(3-u)(3-v) \rangle
\end{aligned}$$

so

$$\begin{aligned}
\iint_{\mathcal{S}_{\text{top}}} \vec{F} \cdot \vec{n}\, dS &= \int_0^3 \int_0^3 \langle 5u, 7v, 11 + uv(3-u)(3-v) \rangle \cdot \langle -(3-2u)v(3-v), -(3-2v)u(3-u), 1 \rangle dudv \\
&= \int_0^3 \int_0^3 -5u(3-2u)v(3-v) - 7v(3-2v)u(3-u) + 11 + uv(3-u)(3-v) dudv
\end{aligned}$$

We can parameterize the bottom surface $\mathcal{S}_{\text{bottom}}$ by $\vec{r}(u,v) = \langle u, v, 0 \rangle$ for $u \in [0,3]$ and $v \in [0,3]$, so

$$\begin{aligned}
\vec{r}_u &= \langle 1, 0, 0 \rangle \\
\vec{r}_v &= \langle 0, 1, 0 \rangle \\
\vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle \\
\vec{F} &= \langle 5u, 7v, 11 \rangle.
\end{aligned}$$

Since $\vec{r}_u \times \vec{r}_v$ has positive z-component but we want the outward normal, we use $-\vec{r}_u \times \vec{r}_v$ and get

$$\iint_{\mathcal{S}_{\text{bottom}}} \vec{F} \cdot \vec{n}\, dS = \int_0^3 \int_0^3 \langle 5u, 7v, 11 \rangle \cdot \langle 0, 0, -1 \rangle dudv = \int_0^3 \int_0^3 -11 dudv$$

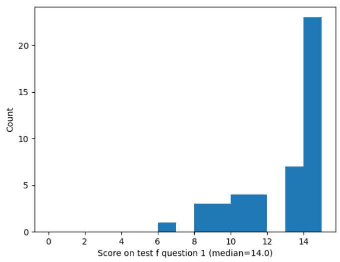
The integral over $\mathcal{S}$ is the sum of these two.

/3 (b) Set up the integral $\iiint_D \text{div}\vec{F}\, dV$. (You do not need to evaluate the integral.)
$\text{div}\vec{F} = 5 + 7 + 1 = 13$, so we have

$$\int_0^3 \int_0^3 \int_0^{xy(3-x)(3-y)} 13 dzdydx$$

/2 (c) Explain how these integrals relate to Green's theorem, Stokes' theorem, and/or the Divergence theorem.
By the (3-dimensional) Divergence theorem, these integrals have the same value.

Scores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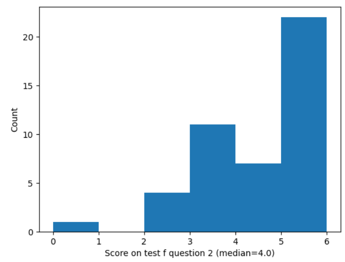
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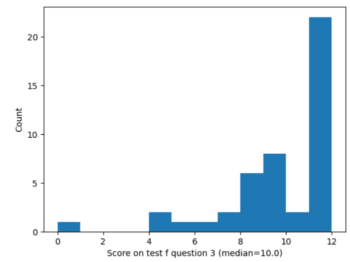
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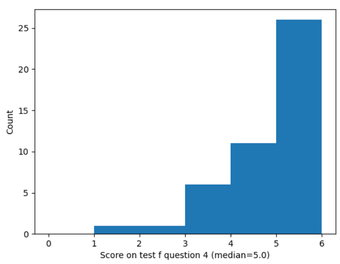
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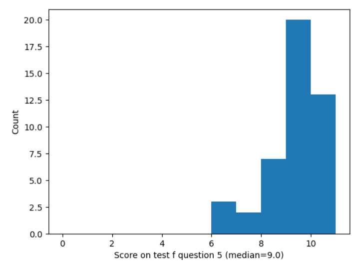
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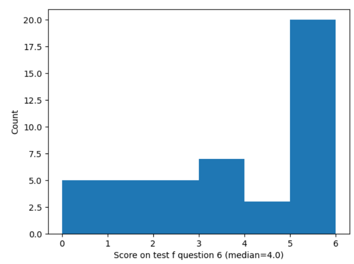
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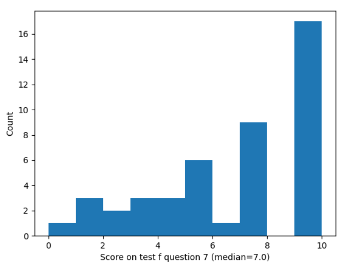
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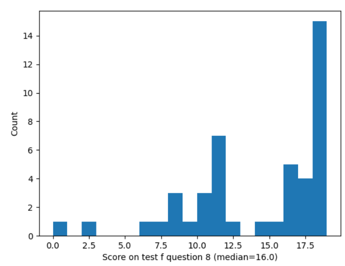
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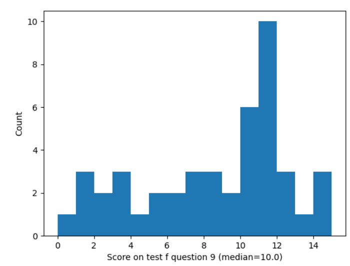
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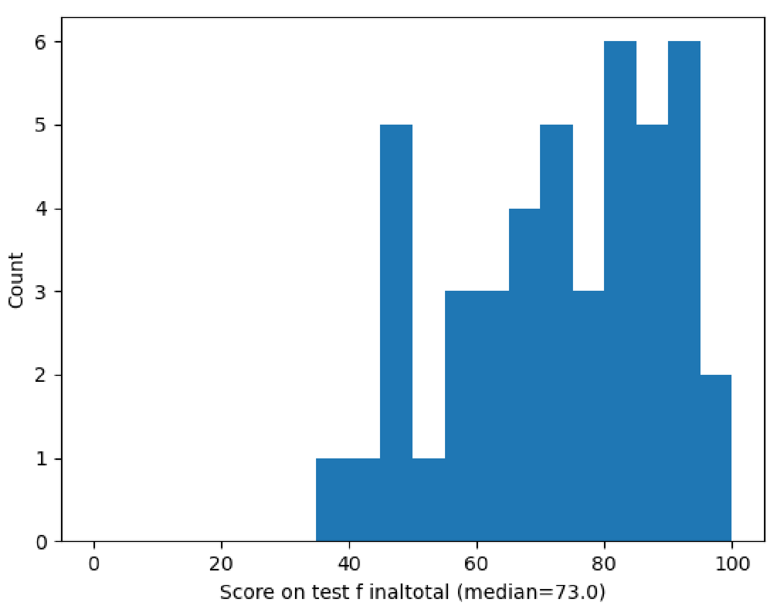
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