score	possible	page
	20	1
	31	2
	24	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/5 1. (a) State the Mean Value Theorem using the template below.

If
- f is continuous on the closed interval $[a, b]$ and
- f is differentiable on the open interval (a, b),

then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

/5 (b) Your friend claims that if $f'(x) > 2$ for all x and $f(0) = 5$, then $f(3) > 12$. Use the Mean Value Theorem to either show that this claim is true or get a different bound on $f(3)$.

Since we are told $f'(x) > 2$ for all x, f must be differentiable everywhere and thus continuous everywhere. Consequently it is continuous on $[0, 3]$ and differentiable on $(0, 3)$, so the assumptions of the Mean Value Theorem are satisfied.

The conclusion of the Mean Value Theorem is that there exists $c \in (0, 3)$ with $f'(c) = \frac{f(3)-5}{3-0}$, so $\frac{f(3)-5}{3} > 2$, which implies $f(3) > 11$. This is the correct bound.

One can show that $f(3) > 12$ is false with the example $f(x) = 2.1x + 5$, which has $f(0) = 5$, $f'(x) = 2.1 > 2$, and $f(3) = 6.3 + 5 = 11.3 < 12$.

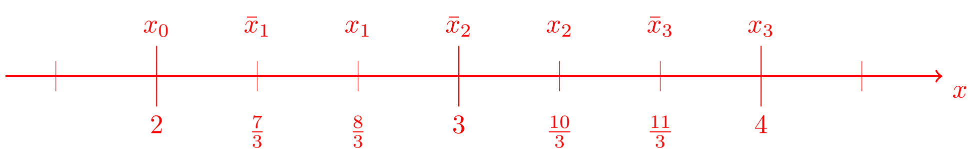
/10 2. Use the Midpoint rule with $n = 3$ to approximate the integral. (Do **not** evaluate the trigonometric functions or otherwise simplify your result.) Include a drawing of your subdivision of the interval and the midpoints used in the approximation.

$$\int_2^4 \sin(\sqrt{x})\, dx$$

The interval $[a, b] = [2, 4]$ has length 2 and we are using 3 rectangles, so the width of each rectangle is $\Delta x = 2/3$.

The base of the first rectangle is $[2, 2 + 2/3]$, which has midpoint $\bar{x}_1 = 2 + 1/3 = 7/3$. The second has base $[2+2/3, 3+1/3]$ and midpoint $\bar{x}_2 = 3$, and the third has base $[3+1/3, 4]$ and midpoint $\bar{x}_3 = 3+2/3 = 11/3$.

We can visualize this as follows.



The area estimate is thus

$$f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x = \sin(\sqrt{7/3})\frac{2}{3} + \sin(\sqrt{3})\frac{2}{3} + \sin(\sqrt{11/3})\frac{2}{3}.$$

3. Find an antiderivative for each function. (The "$+C$" to make it general is not required.)

/3 (a) $f(x) = x^2 \;\Rightarrow F(x) = \frac{x^3}{3} + C$

/3 (b) $f(x) = \dfrac{1}{x^2} \;\Rightarrow F(x) = -x^{-1} + C$

/3 (c) $f(x) = \dfrac{1}{x} \;\Rightarrow F(x) = \ln(x) + C$

/3 (d) $f(x) = \cos(7) \;\Rightarrow F(x) = \cos(7)x + C$

/3 (e) $f(x) = e^x \;\Rightarrow F(x) = e^x + C$

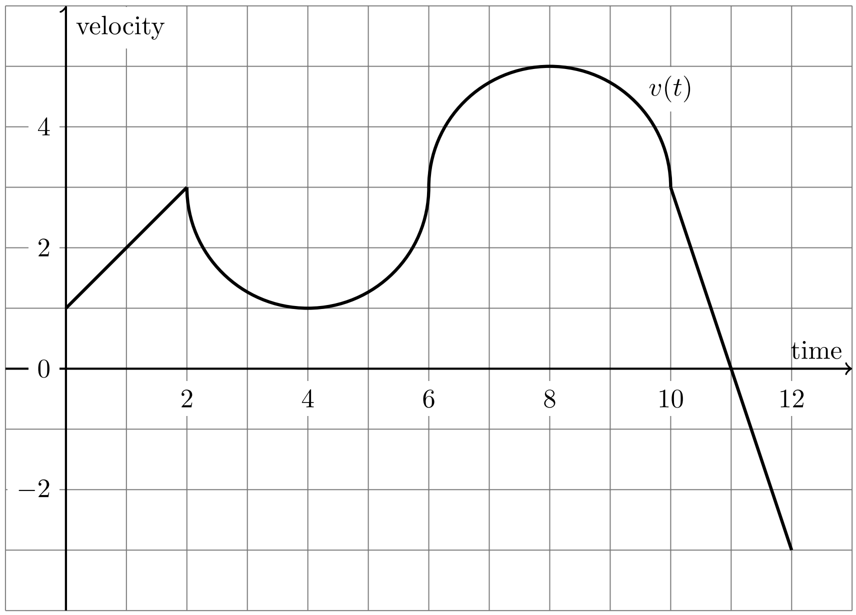
/3 (f) $f(x) = x^2 + 1 \;\Rightarrow F(x) = \frac{x^3}{3} + x + C$

/3 (g) $f(x) = \dfrac{1}{x^2+1} \;\Rightarrow F(x) = \arctan(x) + C$

4. Evaluate each definite integral. (Do not simplify the result.)

/5 (a) $\displaystyle\int_1^2 \cos(x)\,dx = \sin(x)\big|_1^2 = \sin(2) - \sin(1)$

/5 (b) $\displaystyle\int_2^5 \sqrt{x} + 1\,dx = \left(\frac{2}{3}x^{3/2} + x\right)\Bigg|_2^5 = \left(\frac{2}{3}5^{3/2} + 5\right) - \left(\frac{2}{3}2^{3/2} + 2\right)$



5. The graph of a velocity function $v(t)$ on the interval $t \in [0, 12]$ is given above; curved parts are portions of circles. Let $s(t)$ be the position function.

/3 (a) Compute $\int_0^2 v(t)\,dt =$ $\frac{1}{2}(2-0)(1+3) = 4$

/3 (b) Compute $\int_2^{10} v(t)\,dt =$ $3(10-2) = 24$

/3 (c) Compute $\int_{10}^{12} v(t)\,dt =$ 0

/3 (d) Compute the change in position $s(12) - s(0) =$
$\int_0^{12} v(t)\,dt = 4 + 24 + 0 = 28.$

/3 (e) Compute the total distance traveled during the time interval $[0, 12]$.
We need to count the part on the interval $[11, 12]$ as positive rather than negative, so we get $\int_0^{11} v(t)\,dt + \int_{11}^{12} (-v(t))\,dt = 4 + 24 + 3/2 + 3/2 = 31.$

/3 (f) Compute the average value of $v(t)$ on the interval $[0, 12]$.
$$v_{\text{AVG}[0,12]} = \frac{1}{12-0}\int_0^{12} v(t)\,dt = \frac{28}{12} = \frac{7}{3}.$$

/3 (g) On what intervals is $s(t)$ increasing?
$s(t)$ is increasing when $s'(t) = v(t) > 0$, which is on the interval $(0, 11)$.

/3 (h) On what intervals is $s(t)$ concave up?
$s(t)$ is concave up when $s''(t) = v'(t) > 0$, which is on the intervals $(0, 2)$ and $(4, 8)$.

6. For the function $f(x) = x(x-2)^3$:

/2 (a) Find the x- and y-intercepts.

/6 (b) Find the intervals on which f is increasing or decreasing.

/4 (c) Find the local maximum and minimum values of f.

/4 (d) Find the intervals on which f is concave up or concave down.

/4 (e) Find the inflection points.

/5 (f) Use the information above to sketch the graph.

$f(0) = 0$ so the y-intercept is at 0.

Setting $f(x) = x(x-2)^3 = 0$ gives x-intercepts 0 and 2.

We can compute

$$f'(x) = (x-2)^3 + x3(x-2)^2 = ((x-2)+3x)(x-2)^2 = (4x-2)(x-2)^2 = 2(2x-1)(x-2)^2\,.$$

Setting $f'(x) = 0$ then gives critical numbers $1/2$ and 2.

We can compute

$$f''(x) = 2(2)(x-2)^2 + 2(2x-1)2(x-2) = 4((x-2)+(2x-1))(x-2) = 4(3x-3)(x-2) = 12(x-1)(x-2)\,.$$

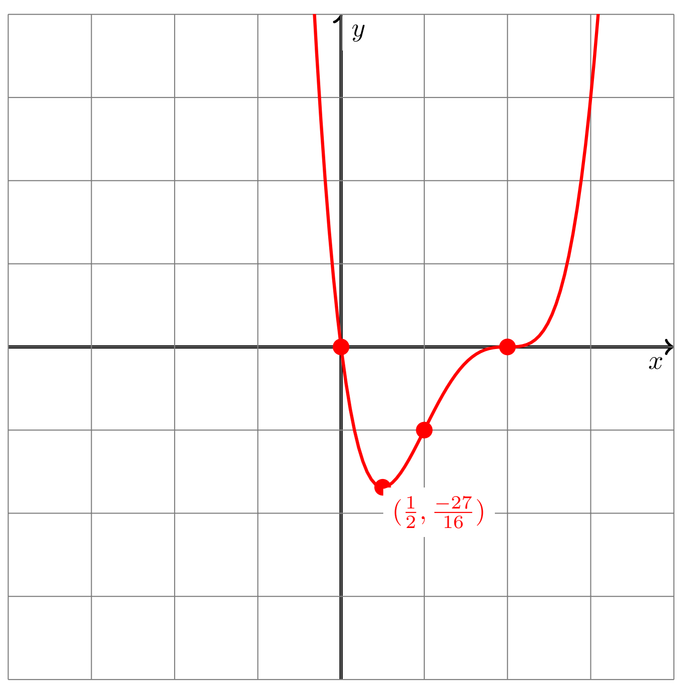
Setting $f'(x) = 0$ then gives possible locations of inflection points at 1 and 2.

Assembling into a chart and checking signs, we have

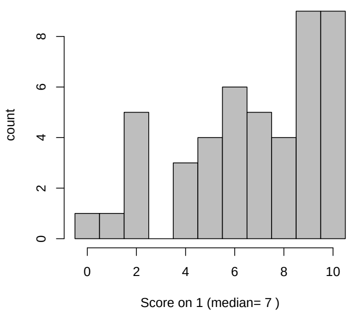
f		min		I.P.		I.P.	
f''	$+$	$+$	$+$	0	$-$	0	$+$
f'	$-$	0	$+$	$+$	$+$	0	$+$
	$(-\infty, 1/2)$	$1/2$	$(1/2, 1)$	1	$(1, 2)$	2	$(2, \infty)$

There is a local minimum at $x = 1/2$ with value $f(1/2) = (1/2)(-3/2)^3 = -27/16$ and no local maxima.

There are inflection points at $(1, f(1)) = (1, -1)$ and $(2, f(2)) = (2, 0)$.



Scores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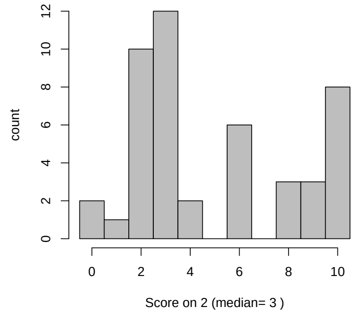
Score on 1 (median= 7)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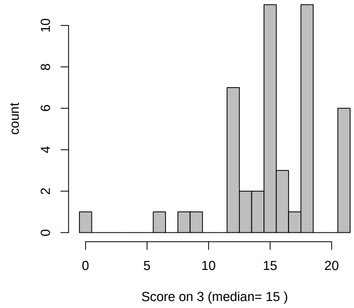
Score on 2 (median= 3)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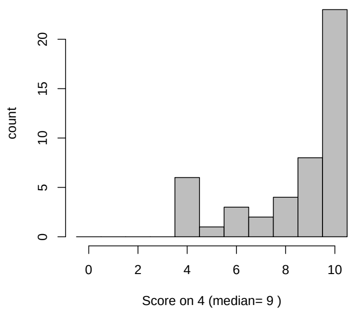
Score on 3 (median= 15)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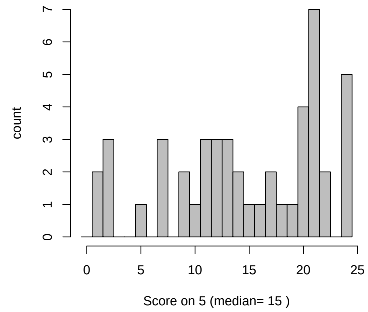
Score on 4 (median= 9)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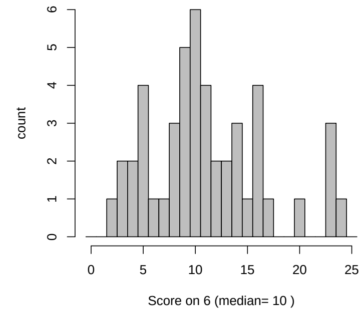
Score on 5 (median= 15)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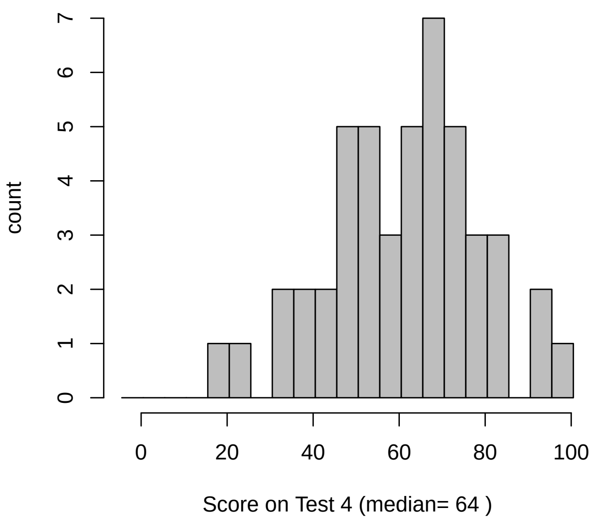
Score on 6 (median= 10)


count
Score on Test 4 (median= 64)