score	possible	page
	25	1
	25	2
	25	3
	25	4
	100	

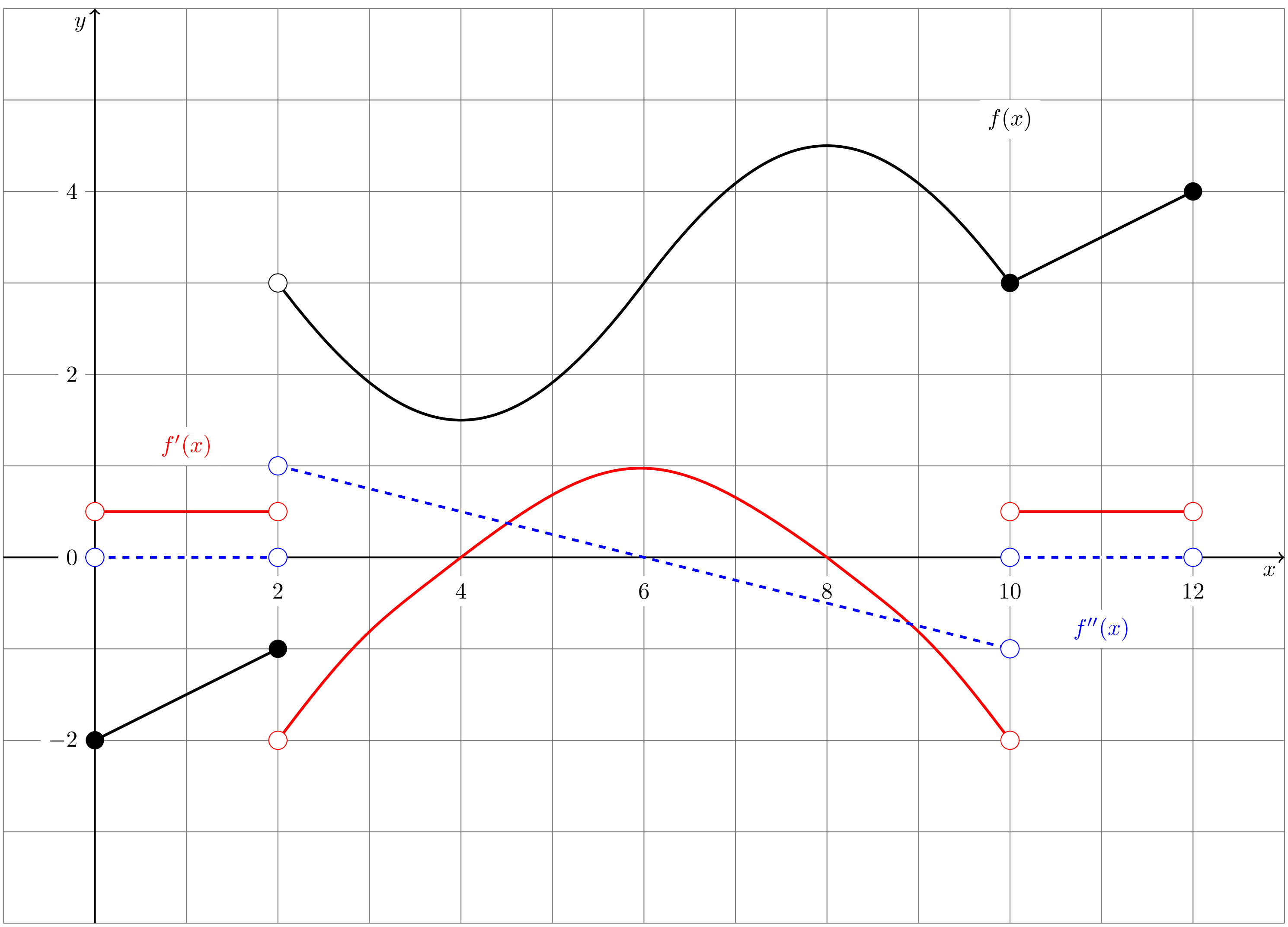
Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. The graph of a function f is given below.

/15 (a) On the same axes, sketch the graph of f'.

/10 (b) On the same axes, using dashed lines or another color, sketch the graph of f''.



2. In this problem, use **the definition of the derivative as a limit** from Chapter 1, not the derivative rules from Chapter 2.

/10 (a) Compute the derivative of $f(x) = 3x^2 + 7$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(3(x+h)^2+7) - (3x^2+7)}{h}$$
$$= \lim_{h\to 0} \frac{(3(x^2+2xh+h^2)+7) - (3x^2+7)}{h} = \lim_{h\to 0} \frac{3x^2+6xh+3h^2+7-3x^2-7}{h}$$
$$= \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} \frac{h(6x+3h)}{h}$$
$$= \lim_{h\to 0}(6x+3h) = 6x\,.$$

/5 (b) Compute the second derivative of $f(x) = 3x^2 + 7$.
We know from above that $f'(x) = 6x$.

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{6(x+h) - 6x}{h}$$
$$= \lim_{h\to 0} \frac{6h}{h} = \lim_{h\to 0} 6 = 6\,.$$

/10 (c) Compute the derivative of $f(x) = (3x-1)^{-1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(3(x+h)-1)^{-1} - (3x-1)^{-1}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{(3(x+h)-1)} - \frac{1}{(3x-1)}}{h} = \lim_{h\to 0} \frac{\frac{(3x-1)-(3(x+h)-1)}{(3x-1)(3(x+h)-1)}}{h}$$
$$= \lim_{h\to 0} \frac{-3h}{h(3x-1)(3(x+h)-1)} = \lim_{h\to 0} \frac{-3}{(3x-1)(3(x+h)-1)}$$
$$= \frac{-3}{(3x-1)(3(x+0)-1)} = \frac{-3}{(3x-1)^2}\,.$$

3. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −1, and no answer is worth +1.

/3 (a) True False $\lim_{x\to 3^-} \frac{x-3}{|x-3|} = 1.$

False. Since $x \to 3^-$ we know $x < 3$ so $x - 3 < 0$ so $|x-3| = -(x-3)$. Thus

$$\lim_{x\to 3^-} \frac{x-3}{|x-3|} = \lim_{x\to 3^-} \frac{x-3}{-(x-3)} = \lim_{x\to 3^-} -1 = -1\,.$$

/3 (b) True False If $\lim_{x\to 3} f(x) = 4$ then $\lim_{x\to 3^+} f(x) = 4$.

True. For the ordinary limit to exist, both one-sided limits must exist and agree with it.

/3 (c) True False If $\lim_{x\to a} f(x) = f(a)$ then f is continuous at a.

True. That is the definition of continuous at a point.

/3 (d) True False If $\lim_{x\to a} f(x) = f(a)$ then f is differentiable at a.

False. That is the definition of continuous at a point, which is necessary but not sufficient to make f differentiable.

/3 (e) True False If $f(x) \le h(x)$, $g(x) \le f(x)$, $h(x) \le 0$, $\lim_{x\to a} g(x) = K$, and $\lim_{x\to a} h(x) = K$, then $\lim_{x\to a} f(x) = K$.

True. This is an awkwardly written version of the Squeeze Theorem. The assumption $h(x) \le 0$ is not needed.

4. In a biology experiment, the number of cells in a test tube was measured at one second intervals to obtain the following data:

Time	0	1	2	3	4	5
Count	1000	1100	1300	1450	1500	1725

/5 (a) What is the average growth rate of the population of cells during this experiment? (Include units.)

During the experiment, the population grew from 1000 to 1723 and time went from 0 to 5 seconds, so the average growth rate is

$$\frac{1725 - 1000}{5 - 0} = \frac{725}{5} = 145\,.$$

The units are cells/s.

/5 (b) Use a central difference to estimate the instantaneous growth rate 3 seconds into the experiment. (Include units.)

Using the central difference formula with $a = 3$ and $h = 1$ yields

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{f(4) - f(2)}{2} = \frac{1500 - 1300}{2} = 100\,.$$

The units are cells/s.

/5 5. (a) State the Intermediate Value Theorem using the template below.

If
- f is continuous on $[a, b]$ and
- $f(a) < N < f(b)$ or $f(a) > N > f(b)$,

then

there exists $c \in (a, b)$ such that $f(c) = N$.

/7 (b) Use the Intermediate Value Theorem to show that the equation $x^7 + x^2 = 4$ has a solution.

Let $f(x) = x^7 + x^2$, so we want to show a solution to $f(x) = 4$ exists. Since x^7 and x^2 are both continuous, so is $f(x)$. Plugging in, we find

$$f(0) = 0 + 0 = 0 < 4 \quad \text{and}$$
$$f(2) = 2^7 + 2^2 = 2^7 + 4 > 4\,.$$

So, by the Intermediate Value Theorem with $a = 0$, $b = 2$, and $N = 4$, there must exist $0 < c < 2$ such that $f(c) = 4$.

6. Suppose f is a function with $f(3) = 5$, $f'(3) = 7$, and $f''(x) < 0$.

/5 (a) Find an equation for the tangent line to f at $x = 3$.

Since the general formula for the tangent line at a is

$$y - f(a) = f'(a)(x - a)\,,$$

here we have

$$y - 5 = 7(x - 3)\,.$$

(In slope-intercept form it is $y = 7x - 16$.)

/5 (b) Use the local linearization of f to estimate $f(2.9)$.

Solving the tangent line equation for y yields the local linearization

$$L(x) = y = 7(x - 3) + 5\,,$$

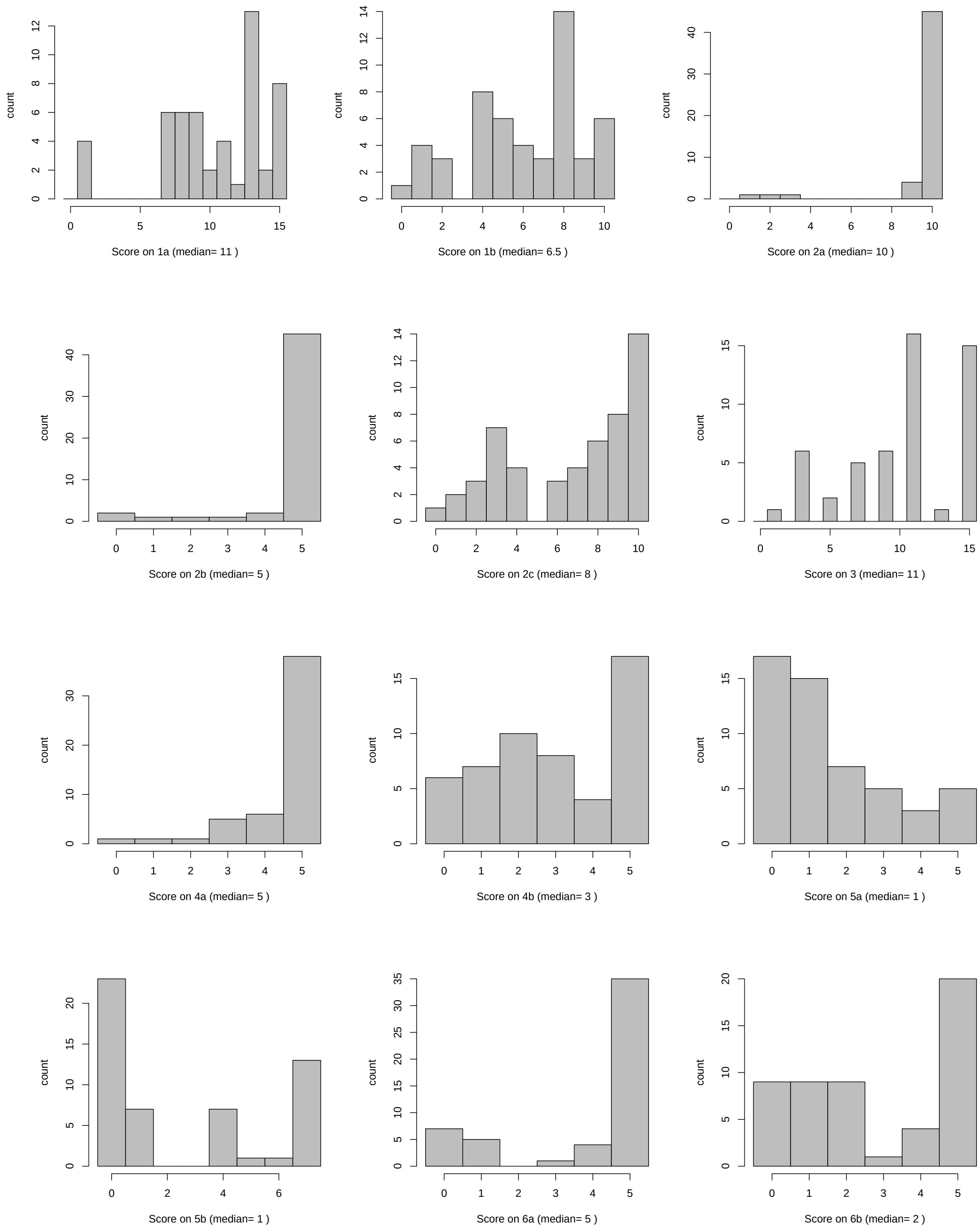
so the approximation is

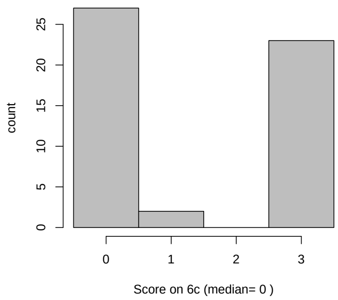
$$f(2.9) \approx L(2.9) = 7(2.9 - 3) + 5 = 7(-0.1) + 5 = -0.7 + 5 = 4.3\,.$$

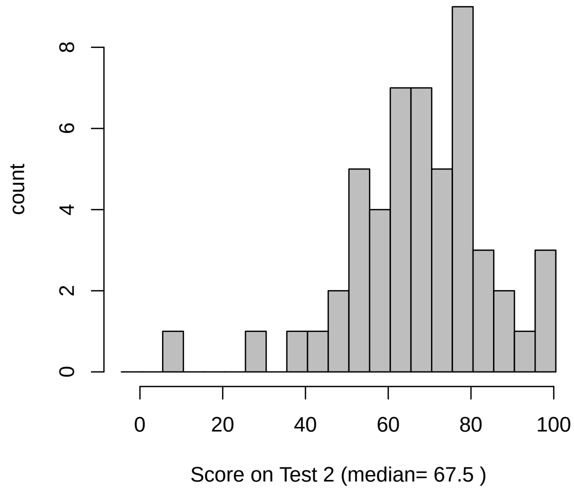
/3 (c) Is your estimate greater than, less than, or equal to $f(2.9)$? If it is not possible to tell from the given information, then say so.

Since $f''(x) < 0$, f is concave down and lies below its tangent line, so the estimate $L(2.9)$ using the tangent line is greater than $f(2.9)$.

Scores


count
Score on 1a (median= 11)
Score on 1b (median= 6.5)
Score on 2a (median= 10)
Score on 2b (median= 5)
Score on 2c (median= 8)
Score on 3 (median= 11)
Score on 4a (median= 5)
Score on 4b (median= 3)
Score on 5a (median= 1)
Score on 5b (median= 1)
Score on 6a (median= 5)
Score on 6b (median= 2)


count
0
5
10
15
20
25
0
1
2
3
Score on 6c (median= 0)


count
0
2
4
6
8
0
20
40
60
80
100
Score on Test 2 (median= 67.5)