score	possible	page
	25	1
	30	2
	20	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −1, and no answer is worth +1.

/3 (a) True False $\lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty$.

[3.7#9] True. $\lim_{x\to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$.

/3 (b) True False A local maximum of a function $f(x)$ can only occur at a point where $f'(x) = 0$.

False. It could also occur where $f'(x)$ does not exist.

/3 (c) True False If $f'(x) = g'(x)$ then $f(x) = g(x)$.

False. If $f(x) = g(x) + 5$ then $f'(x) = g'(x)$. See section 4.2 Corollary 7.

/3 (d) True False If $f'(x) = 2$ and $f(3) = 5$ then $f(x) = 2x - 1$.

True. Antidifferentiating gives $f(x) = 2x + C$ and plugging in 3 gives $5 = f(3) = 2\cdot 3 + C = 6 + C$ so $C = -1$ and $f(x) = 2x - 1$.

/3 (e) True False If $f'(x)$ exists and is nonzero for all x then $f(0) \neq f(1)$.

[Chapter 4 review True/False #19] True. Since $f'(x)$ exists we know f is differentiable and so continuous. If $f(0) = f(1)$ then Rolle's theorem implies there is $c \in (0, 1)$ with $f'(c) = 0$. Since we know $f'(x) \neq 0$, there is a contradiction, so $f(0) \neq f(1)$.

/10 2. Use Newton's method with the initial approximation $x_1 = 2$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Leave the answer as a fraction.

[Similar to 4.6 #6-8] Setting $f(x) = x^3 + x + 3$ gives $f'(x) = 3x^2 + 1$. Newton's method gives the next approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 + 2 + 3}{3(2^2) + 1} = 2 - \frac{13}{13} = 1\,.$$

3. For each $f'(x)$, find an $f(x)$ that has that derivative.
(Each $f(x)$ should have "$+C$", but you do not have to include it.)

/2 (a) $f'(x) = 1 \quad \Rightarrow \quad f(x) = x$

/2 (b) $f'(x) = \sin(2) \quad \Rightarrow \quad f(x) = \sin(2)x$

/2 (c) $f'(x) = 3x \quad \Rightarrow \quad f(x) = \frac{3}{2}x^2$

/2 (d) $f'(x) = 4x^{-1} \quad \Rightarrow \quad f(x) = 4\ln(x)$

/2 (e) $f'(x) = 5x^{-2} \quad \Rightarrow \quad f(x) = -5x^{-1}$

/2 (f) $f'(x) = 6^x \quad \Rightarrow \quad f(x) = 6^x/\ln(6)$

/2 (g) $f'(x) = 7\sin(x) \quad \Rightarrow \quad f(x) = -7\cos(x)$

/2 (h) $f'(x) = 8\cosh(x) \quad \Rightarrow \quad f(x) = 8\sinh(x)$

/2 (i) $f'(x) = \dfrac{9}{1+x^2} \quad \Rightarrow \quad f(x) = 9\tan^{-1}(x)$

/2 (j) $f'(x) = 10x + 11 \quad \Rightarrow \quad f(x) = 5x^2 + 11x$

/2 (k) $f'(x) = \dfrac{1}{11\sqrt{1-x^2}} \quad \Rightarrow \quad f(x) = \frac{1}{11}\sin^{-1}(x)$

/2 (l) $f'(x) = 12\dfrac{1}{x\sqrt{x^2-1}} \quad \Rightarrow \quad f(x) = 12\sec^{-1}(x)$

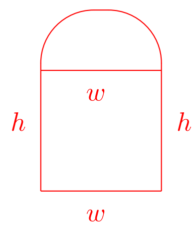
/2 (m) $f'(x) = 13\sec(x)\tan(x) \quad \Rightarrow \quad f(x) = 13\sec(x)$

/2 (n) $f'(x) = 14\sec^2(x) \quad \Rightarrow \quad f(x) = 14\tan(x)$

/2 (o) $f'(x) = x^{15/16} \quad \Rightarrow \quad f(x) = \frac{16}{31}x^{31/16}$

/20 4. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 10 m, find the dimensions of the window so that the greatest possible amount of light is admitted.

[4.4 # 25 with different perimeter.]



With the diagram above, the perimeter is $2h + w + \pi w/2 = 2h + (1 + \pi/2)w$. Setting equal to 10 m gives the constraint $2h + (1 + \pi/2)w = 10\,\text{m}$. Solving for h gives $h = 5\,\text{m} - \frac{1+\pi/2}{2}w$.

We want to maximize the area $A = wh + \pi(w/2)^2/2 = wh + \frac{\pi}{8}w^2$. Substituting in for h gives

$$A = w\left(5\,\text{m} - \frac{1+\pi/2}{2}w\right) + \frac{\pi}{8}w^2 = 5w\,\text{m} + \left(-\frac{1+\pi/2}{2} + \frac{\pi}{8}\right)w^2 = 5w\,\text{m} - \frac{1+\pi/4}{2}w^2\,.$$

Differentiating yields

$$A' = 5\,\text{m} - (1 + \pi/4)\,w\,,$$

which is 0 when $w = 5\,\text{m}/(1 + \pi/4) = 20\,\text{m}/(4 + \pi)$. The second derivative is $A'' = -(1 + \pi/4) < 0$ so this w gives a maximum. Plugging into the constraint, we have

$$h = 5\,\text{m} - \frac{1+\pi/2}{2}\frac{20\,\text{m}}{4+\pi} = \frac{5(4+\pi) - 10(1+\pi/2)}{4+\pi}\,\text{m} = \frac{10}{4+\pi}\,\text{m}\,.$$

Therefore the window should be $20/(4 + \pi)$ m wide and the straight part should be $10/(4 + \pi)$ m tall. (Including the semicircle it is $h + w/2 = 20/(4 + \pi)$m tall.)

5. For the function $f(x) = \dfrac{x^2}{x^2-9}$, which has $f'(x) = \dfrac{-18x}{(x^2-9)^2}$ and $f''(x) = \dfrac{54(x^2+3)}{(x^2-9)^3}$:

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/4 (c) Find the intervals on which f is increasing or decreasing.

/4 (d) Find the local maximum and minimum values of f.

/6 (e) Find the intervals of concavity and the inflection points .

/5 (f) Use the information above to sketch the graph.

[Similar to 4.3 # 11, 12]

(f has even symmetry, so we could save half the work, but this is optional.)

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is 0 and there are vertical asymptotes at $x = -3$ and $x = 3$.

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-9} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = \lim_{x\to\pm\infty} \frac{1}{1} = 1$$

so there is a horizontal asymptote at $y = 1$.

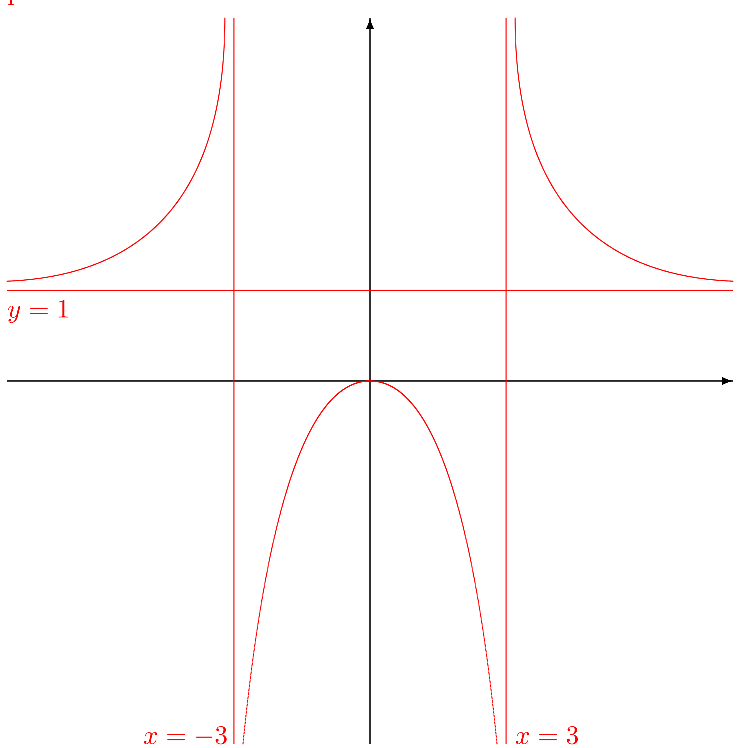
The given $f'(x) = \dfrac{-18x}{(x^2-9)^2}$ is zero at $x = 0$ and does not exist at the vertical asymptotes $x = -3$ and $x = 3$.

The given $f''(x) = \dfrac{54(x^2+3)}{(x^2-9)^3}$ is never zero and does not exist at the vertical asymptotes $x = -3$ and $x = 3$.

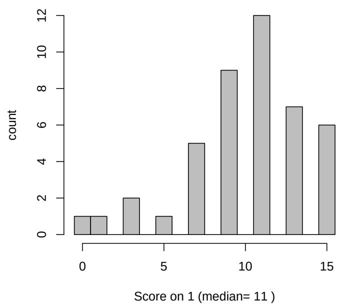
Assembling into a chart and checking signs, we have

f		V.A		$\rightarrow$		V.A.	
f''	$+$	DNE	$-$	$-$	$-$	DNE	$+$
f'	$+$	DNE	$+$	0	$-$	DNE	$-$
	$(-\infty, -3)$	-3	$(-3, 0)$	0	$(0, 3)$	3	$(3, \infty)$

There is a local maximum at $x = 0$ with value $f(0) = 0$ and no local minima. There are no inflection points.



Scores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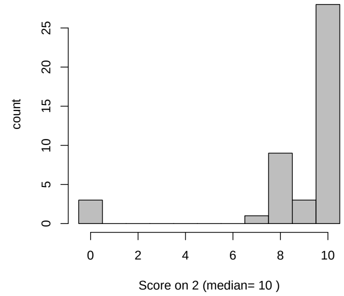
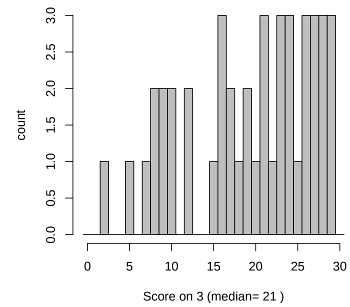
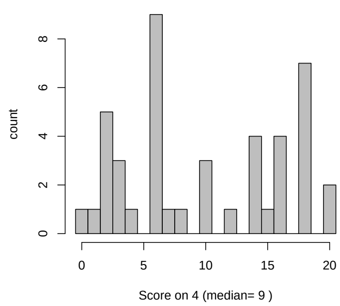
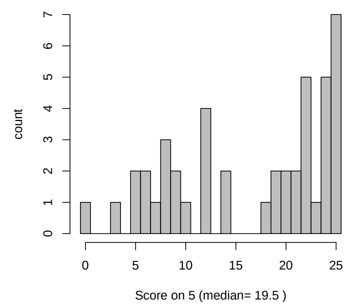
Score on 1 (median= 11)


count
Score on 2 (median= 10)


count
Score on 3 (median= 21)


count
Score on 4 (median= 9)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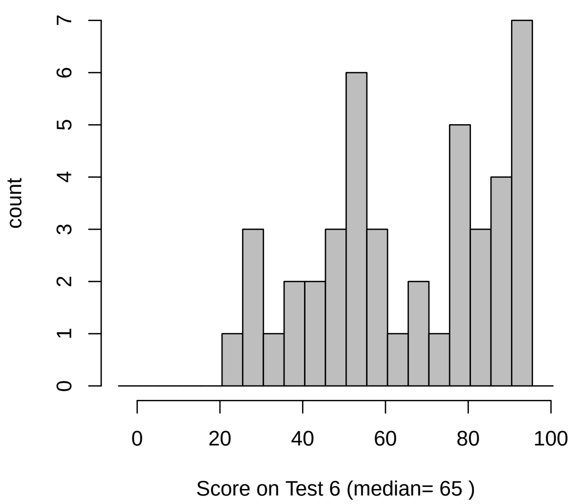
Score on 5 (median= 19.5)


count
Score on Test 6 (median= 65)