score	possible	page
	25	1
	20	2
	30	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −1, and no answer is worth +1.

/3 (a) True False If $f(a) < N < f(b)$ then there exists $c \in (a, b)$ such that $f(c) = N$.
False. This is almost the Intermediate Value Theorem but lacks the assumption that f is continuous on $[a, b]$.

/3 (b) True False $\lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty$.
[3.7#9] True. $\lim_{x\to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$.

/3 (c) True False A local maximum of a function $f(x)$ can only occur at a point where $f'(x) = 0$.

False. It could also occur where $f'(x)$ does not exist.

/3 (d) True False If f is a continous function on the interval $[a, b]$ then f attains an absolute maximum at some number c in $[a, b]$.
True. This is the Extreme Value Theorem in section 4.1.

/3 (e) True False If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).
True. This is Theorem 5 in section 4.2 and is a consequence of the Mean Value Theorem.

/3 (f) True False If $f'(x) > 2$ for all x and $f(0) = 5$, then $f(3) > 11$.
[Similar to 4.3 #23, 25] True. The Mean Value Theorem implies there exists $c \in (0, 3)$ with $f'(c) = \frac{f(3)-5}{3-0}$, so $\frac{f(3)-5}{3} > 2$, which implies $f(3) > 11$. (The simpler logic of saying f must be above the line through $(0, 5)$ with slope 2 also works.) .

/7 2. For the function $f(x) = x^{1/3}(x + 4)$, find all c such that $f''(c) = 0$ or $f''(c)$ does not exist.
[Similar to part of 4.3 #33] We can compute

$$f'(x) = (1/3)x^{-2/3}(x + 4) + x^{1/3} = (1/3)x^{-2/3}((x + 4) + 3x) = (4/3)x^{-2/3}(x + 1) \quad \text{and}$$
$$f''(x) = (4/3)\left((-2/3)x^{-5/3}(x + 1) + x^{-2/3}\right) = (4/9)x^{-5/3}((-2)(x + 1) + 3x) = (4/9)x^{-5/3}(x - 2).$$

$f''(c) = 0$ when $c = 2$ and $f''(c)$ DNE when $c = 0$.

3. Compute the limits. Show your work and/or explain your reasoning.

/5 (a) $\lim_{x\to\infty} x^2 e^{-x^2} =$

[Similar to 3.7 #25] Plugging in yields $\infty \cdot 0$, which is indeterminate but not in the form for L'Hôpital's rule. Rewriting to get ∞/∞ form, we can apply L'Hôpital's rule to get

$$\lim_{x\to\infty} \frac{x^2}{e^{x^2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0\,.$$

/5 (b) $\lim_{x\to 0} 3x^2 \sin\left(\frac{1}{x}\right) =$

[Similar to 1.4 Example 9] Set

$$f(x) = -3x^2\,,$$
$$g(x) = 3x^2 \sin\left(\frac{1}{x}\right)\,, \quad \text{and}$$
$$h(x) = 3x^2\,.$$

Since $|\sin(\cdot)| \le 1$, we have $f(x) \le g(x) \le h(x)$ when x is near 0 (and for all $x \ne 0$). We can compute $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so the assumptions of the Squeeze Theorem are satisfied and we can conclude $\lim_{x\to 0} g(x) = 0$.

/5 4. (a) State the Mean Value Theorem (MVT) using the template below.

If
- f is continuous on the closed interval $[a, b]$ and
- f is differentiable on the open interval (a, b),

then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

/5 (b) Verify that the function $f(x) = 1/x$ satisfies each of the hypotheses of the MVT on the interval $[1, 3]$. Then find all numbers c that satisfy the conclusion of the MVT.

[4.2 #12] $f(x) = 1/x$ is differentiable on $(0, \infty)$ and hence differentiable on $(1, 3)$ and continuous on $[1, 3]$.

We have $f'(x) = -x^{-2}$ and $\frac{f(3)-f(1)}{3-1} = \frac{\frac{1}{3}-1}{2} = -\frac{1}{3}$. Setting $-x^{-2} = -\frac{1}{3}$ and solving gives $x = \pm\sqrt{3}$, of which $\sqrt{3} \in (1, 3)$.

/10 5. Find the absolute maximum and minimum values of $f(x) = x^3 - 9x^2 + 1$ on the interval $[-2, 2]$. [Similar to 4.1 #40] $f'(x) = 3x^2 - 18x = 3x(x-6)$ so the critical numbers are $x = 0$ and $x = 6$. Since $x = 6$ is not in the interval, we discard it. Evaluating at $x = 0$ and at the endpoints we get

$$f(0) = 1\,,$$
$$f(-2) = (-2)^3 - 9(-2)^2 + 1 = -8 - 36 + 1 = -43\,, \quad \text{and}$$
$$f(2) = 2^3 - 9 \cdot 2^2 + 1 = 8 - 36 + 1 = -27\,.$$

Thus the absolute maximum is 1 and occurs at $x = 0$ and the absolute minimum is -43 and occurs at $x = -2$.

6. Sketch the graph of a single function f that has all of the following properties:

/4 (a) $f'(1) = f'(-1) = 0$.

/2 (b) $f'(x) < 0$ if $|x| < 1$.

/4 (c) $f'(x) > 0$ if $1 < |x| < 2$.

/4 (d) $f'(x) = -1$ if $|x| > 2$.

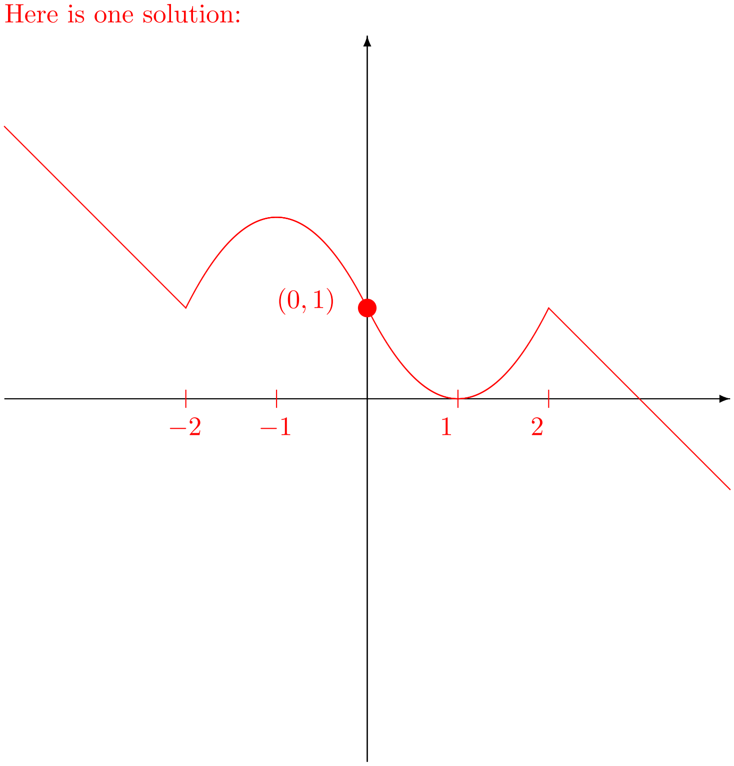
/3 (e) $f''(x) < 0$ if $-2 < x < 0$.

/3 (f) An inflection point at $(0, 1)$.

[4.3 #20] Since $f'(x) = -1$ if $|x| > 2$, we know $f''(x) = 0$ if $|x| > 2$. Since there is an inflection point at $(0, 1)$ and $f''(x) < 0$ for $-2 < x < 0$, we know $f''(x) > 0$ for $0 < x <?$. Organizing into a chart, we have

f	↘		◜	→	◝	↘, IP	◟	→	↗		↘
f''	0		$-$	$-$	$-$	?	$+, ?$				0
f'	-1		$+$	0	$-$	$-$	$-$	0	$+$		-1
	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, \infty)$

It is consistent if we make $f''(x) > 0$ if $0 < x < 2$. At -2 and 2 there must be a cusp or discontinuity. Here is one solution:



7. For the function $f(x) = \dfrac{x}{x^2+4}$, which has $f'(x) = \dfrac{(2+x)(2-x)}{(x^2+4)^2}$ and $f''(x) = \dfrac{2x(x+2\sqrt{3})(x-2\sqrt{3})}{(x^2+4)^3}$:

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/3 (c) Find the intervals on which f is increasing or decreasing.

/4 (d) Find the local maximum and minimum values of f.

/7 (e) Find the intervals of concavity and the inflection points .

/5 (f) Use the information above to sketch the graph.

(f has odd symmetry, so we could save half the work, but this is optional.)

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is never 0 so there are no vertical asymptotes.

$\lim_{x\to\pm\infty} \frac{x}{x^2+4} = \lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

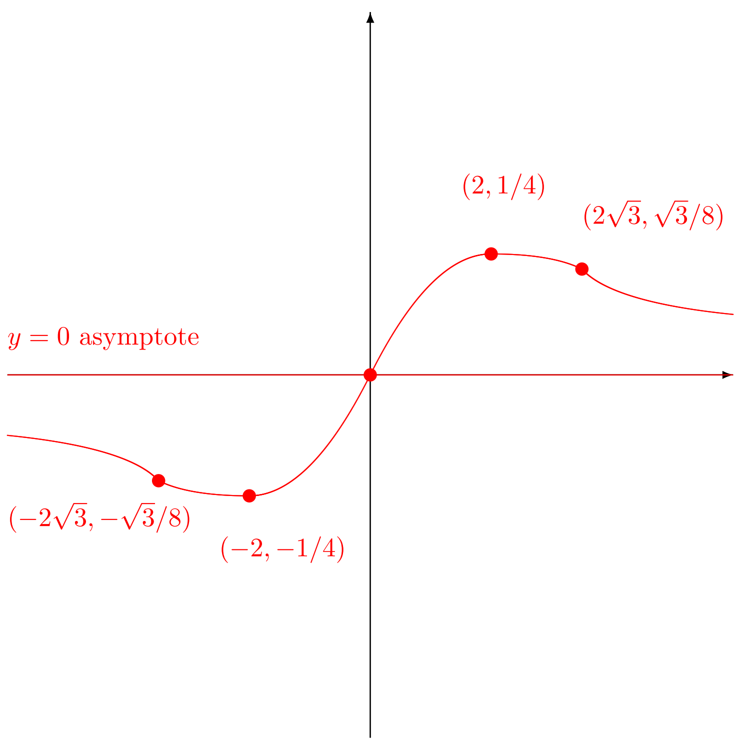
The given $f'(x) = \dfrac{(2+x)(2-x)}{(x^2+4)^2}$ is zero at $x = -2$ and $x = 2$.

The given $f''(x) = \dfrac{2x(x+2\sqrt{3})(x-2\sqrt{3})}{(x^2+4)^3}$ is 0 at $x = 0$, $x = -2\sqrt{3}$, and $x = 2\sqrt{3}$.

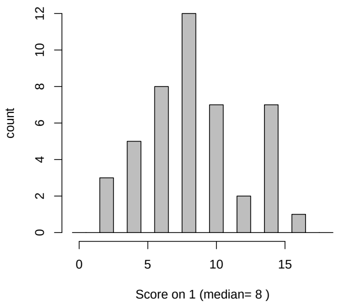
Assembling into a chart and checking signs, we have

f	╮	I.P.	╰	→	╯	I.P.	╭	→	╮	I.P.	╰
f''	−	0	+	+	+	0	−	−	−	0	+
f'	−	−	−	0	+	+	+	0	−	−	−
	$(-\infty, -2\sqrt{3})$	$-2\sqrt{3}$	$(-2\sqrt{3}, -2)$	-2	$(-2, 0)$	0	$(0, 2)$	2	$(2, 2\sqrt{3})$	$2\sqrt{3}$	$(2\sqrt{3}, \infty)$

There is a local max at $x = 2$ with value $f(2) = 2/(2^2+4) = 1/4$ and a local min at $x = -2$ with value $f(-2) = -2/(2^2+4) = -1/4$. There are inflection points at $(-2\sqrt{3}, f(-2\sqrt{3})) = (-2\sqrt{3}, -2\sqrt{3}/(12+4)) = (-2\sqrt{3}, -\sqrt{3}/8)$, $(0, 0)$, and $(2\sqrt{3}, f(2\sqrt{3})) = (2\sqrt{3}, \sqrt{3}/8)$.



Scores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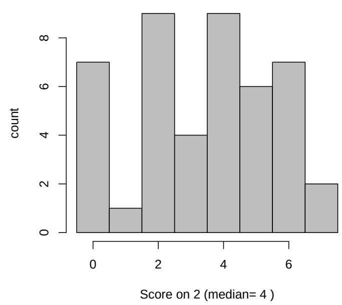
Score on 1 (median= 8)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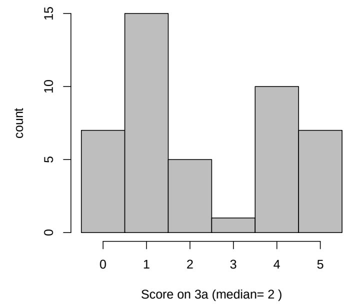
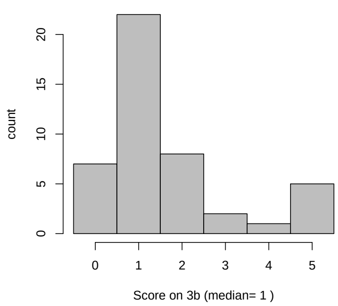
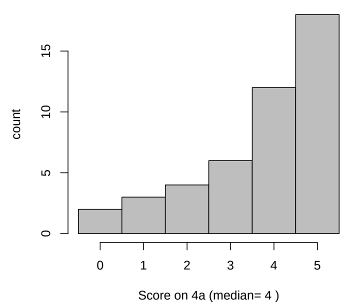
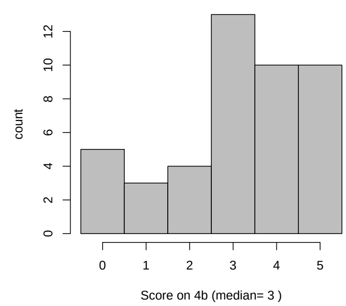
Score on 2 (median= 4)


count
Score on 3a (median= 2)


count
Score on 3b (median= 1)


count
Score on 4a (median= 4)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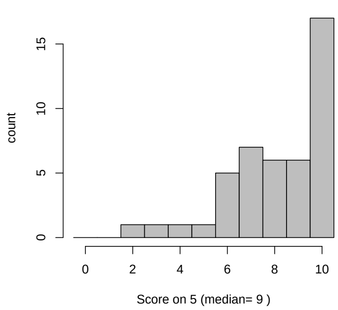
Score on 4b (median= 3)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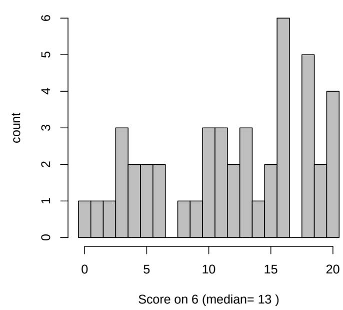
Score on 5 (median= 9)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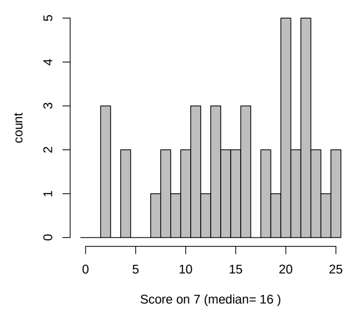
Score on 6 (median= 13)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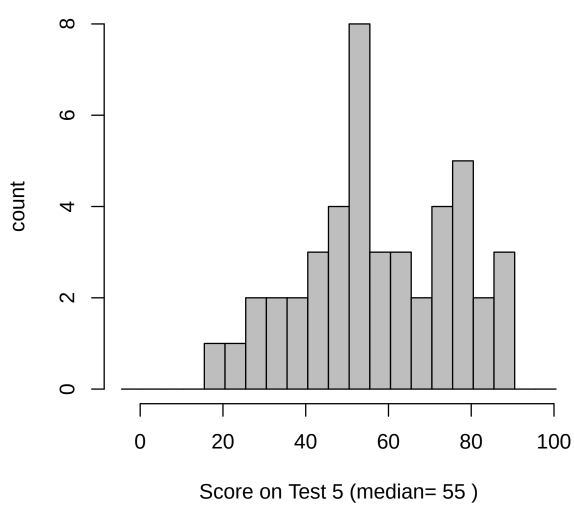
Score on 7 (median= 16)


count
Score on Test 5 (median= 55)