score	possible	page
	30	1
	30	2
	20	3
	20	4
	100	

Name: ____________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.

/3 (a) True False $\sinh(0) + \cosh(0) = 0$.
False. $\sinh(0) + \cosh(0) = \frac{1}{2}(e^0 - e^0) + \frac{1}{2}(e^0 + e^0) = 0 + 1 = 1 \neq 0$.

/3 (b) True False If $f(a) < N < f(b)$ then there exists $c \in (a, b)$ such that $f(c) = N$.
False. This is almost the Intermediate Value Theorem but lacks the assumption that f is continuous on $[a, b]$.

/3 (c) True False $\lim_{x\to 0^+} \frac{2}{x} e^{3x} = \infty$.
True. $\lim_{x\to 0^+} \frac{2}{x} e^{3x} = 2e^0 \lim_{x\to 0^+} \frac{1}{x} = 2\infty = \infty$.

/3 (d) True False If a function has a horizontal asymptote, then the graph of the function cannot cross the horizontal asymptote.
False. The horizontal asymptote only tells us about $\lim_{x\to -\infty} f(x)$ or $\lim_{x\to\infty} f(x)$, not what happens in between.

/3 (e) True False $\lim_{x\to\infty} \ln(5x^9) = \infty$.
True. $\lim_{x\to\infty} \ln(5x^9) = \lim_{t\to\infty} \ln(t) = \infty$.

/3 (f) True False $\lim_{x\to\infty} \arctan(e^x) = \frac{\pi}{2}$.
[3.5#37] True. $\lim_{x\to\infty} \arctan(e^x) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$.

/3 (g) True False $\lim_{x\to 0^+} \frac{\ln(x)}{x} = \infty$.
[3.7#9] False. $\lim_{x\to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty \neq \infty$.

/3 (h) True False $\frac{d}{dx}(\mathrm{arcsec}(x)) = \frac{1}{x\sqrt{x^2-1}}$.
True. See table 11 in section 3.5.

/3 (i) True False $\frac{d}{dy}\left(\tan^{-1}(y^2)\right) = \frac{2y}{1+y^4}$.
True. Using the chain rule, $\frac{d}{dy}\left(\tan^{-1}(y^2)\right) = \frac{1}{1+(y^2)^2}(2y) = \frac{2y}{1+y^4}$.

/3 (j) True False $\frac{d}{dx}(\sinh(\cosh(x))) = \cosh(\cosh(x))\sinh(x)$.
True. Using the chain rule, $\frac{d}{dx}(\sinh(\cosh(x))) = \sinh'(\cosh(x))\cosh'(x) = \cosh(\cosh(x))\sinh(x)$.

2. Compute the following limits. Show your work and/or explain your reasoning. In particular, say when you use L'Hôpital's rule, and give the functions used to squeeze when you use the Squeeze Theorem.

/6 (a) $\lim_{t\to 0} \dfrac{e^{2t}-1}{\sin(t)} =$

[Similar to 3.7#5] Plugging in gives $\frac{e^0-1}{0} = \frac{0}{0}$, which is indeterminate in the correct form to apply L'Hôpital's rule, so we apply it and get

$$\lim_{t\to 0} \frac{e^{2t}2}{\cos(t)} = \frac{1\cdot 2}{1} = 2\,.$$

/6 (b) $\lim_{x\to 0} \dfrac{x-\sin(x)}{x^3} =$

[Based on 3.7#21] Plugging in keeps giving 0/0 indeterminate form, so we apply L'Hôpital's rule 3 times, to get

$$\begin{aligned} &= \frac{0-0}{0} = \frac{0}{0} \quad \text{indeterminate} \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{1-\cos(x)}{3x^2} = \frac{0}{0} \quad \text{indeterminate} \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{\sin(x)}{6x} = \frac{0}{0} \quad \text{indeterminate} \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}\,. \end{aligned}$$

(Instead of the last use of L'Hôpital's rule, we could have used the known limit $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$.)

/6 (c) $\lim_{x\to 0^+} x\ln(x) =$

[3.7 Example 6] Plugging in yields $0^+(-\infty)$, which is indeterminate but not in the form for L'Hôpital's rule. Rewriting to get $-\infty/\infty$ form, we can apply L'Hôpital's rule to get

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0\,.$$

/6 (d) $\lim_{x\to\infty} \dfrac{3x}{\sqrt{x^2+1}} =$

[Similar to 3.7#43] Although plugging in gives ∞/∞ indeterminate form, L'Hôpital's rule gets stuck in a loop. Instead we can use algebra and get

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+1}}\,\frac{x^{-1}}{x^{-1}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+x^{-2}}} = \frac{3}{\sqrt{1+0}} = 3\,.$$

/6 (e) $\lim_{x\to 0} 5x^2\cos\left(\dfrac{1}{x}\right) =$

[Similar to 1.4 Example 9] Set

$$\begin{aligned} f(x) &= -5x^2\,, \\ g(x) &= 5x^2\cos\left(\frac{1}{x}\right)\,, \quad \text{and} \\ h(x) &= 5x^2\,. \end{aligned}$$

Since $|\cos(\cdot)| \le 1$, we have $f(x) \le g(x) \le h(x)$ when x is near 0 (and for all $x \neq 0$). We can compute $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so the assumptions of the Squeeze Theorem are satisfied and we can conclude $\lim_{x\to 0} g(x) = 0$.

/10 3. Use a linear approximation to estimate $(8.03)^{2/3}$.

[Similar to 2.8#13] Set $f(x) = x^{2/3}$ so $f'(x) = (2/3)x^{-1/3}$. Selecting $a = 8$ we have the linear approximation

$$f(x) \approx L_8(x) = f(8) + f'(8)(x-8) = 8^{2/3} + \frac{2}{3}8^{-1/3}(x-8) = 4 + \frac{1}{3}(x-8)$$

so $(8.03)^{2/3} = f(8.03) \approx 4 + \frac{0.03}{3} = 4.01$.

/10 4. Use logarithmic differentiation to compute the derivative of $y = \dfrac{\left(1+x^6\right)^{\sin(2x)}}{\sqrt{x^4+1}}$.

Applying the natural logarithm to both sides and using rules of logarithms yields

$$\ln(y) = \sin(2x)\ln(1+x^6) - \frac{1}{2}\ln(x^4+1)\,.$$

Differentiating both sides yields

$$\frac{y'}{y} = \cos(2x)2\ln(1+x^6) + \sin(2x)\frac{6x^5}{1+x^6} - \frac{1}{2}\frac{4x^3}{x^4+1}\,.$$

Solving for y' then gives

$$y' = \left(\cos(2x)2\ln(1+x^6) + \sin(2x)\frac{6x^5}{1+x^6} - \frac{1}{2}\frac{4x^3}{x^4+1}\right)\frac{\left(1+x^6\right)^{\sin(2x)}}{\sqrt{x^4+1}}\,.$$

/10 5. Let $f(x) = 4 + x^2 + \tan\left(\frac{\pi x}{2}\right)$ on the domain $-1 < x < 1$. Find $\left(f^{-1}\right)'(4)$.

[Similar to 3.2#37] We will use the general formula

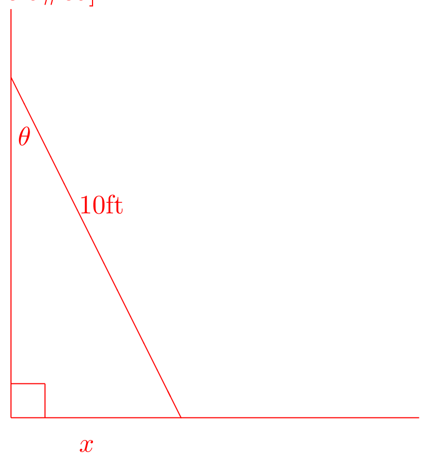
$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}\,.$$

By observing that $f(0) = 4 + 0^2 + \tan(0) = 4$, we have $f^{-1}(4) = 0$. We can compute $f'(x) = 2x + \sec^2(\pi x/2)\frac{\pi}{2}$, so

$$\left(f^{-1}\right)'(4) = \frac{1}{f'\left(f^{-1}(4)\right)} = \frac{1}{f'(0)} = \frac{1}{0 + \sec^2(0)\frac{\pi}{2}} = \frac{2\cos^2(0)}{\pi} = \frac{2}{\pi}\,.$$

/10 6. A ladder 10 ft long is leaning against a vertical wall. It starts slipping, such that the bottom of the ladder slides away from the base of the wall at a speed of 3ft/s. Draw and label a diagram illustrating this scenario. How fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

[Similar to 3.5#39]



We know $\frac{dx}{dt} = 3\text{ft/s}$ and want to know $\frac{d\theta}{dt}$. We can relate x and θ by $\theta = \arcsin(x/(10\text{ft}))$. Differentiating yields

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - ((x/(10\text{ft}))^2}}\frac{1}{10\text{ft}}\frac{dx}{dt}\,.$$

Inserting $\frac{dx}{dt} = 3\text{ft/s}$ and $x = 6\text{ft}$ yields

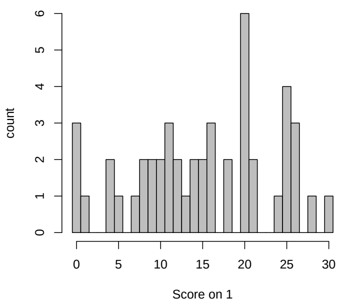
$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - ((6\text{ft}/(10\text{ft}))^2}}\frac{1}{10\text{ft}}\frac{3\text{ft}}{\text{s}} = \frac{1}{\sqrt{1 - (3/5)^2}}\frac{3}{10\text{s}} = \frac{1}{\sqrt{16/25}}\frac{3}{10\text{s}} = \frac{5}{4}\frac{3}{10\text{s}} = \frac{3}{8\text{s}}\,.$$

Alternatively, you could use $\sin(\theta) = x/(10\text{ft})$ and implicitly differentiate to get $\cos(\theta)\frac{d\theta}{dt} = \frac{1}{10\text{ft}}\frac{dx}{dt}$ and note $\cos(\theta) = \sqrt{(10\text{ft})^2 - x^2}/(10\text{ft})$. We then have

$$\frac{d\theta}{dt} = \frac{10\text{ft}}{\sqrt{(10\text{ft})^2 - x^2}}\frac{1}{10\text{ft}}\frac{dx}{dt} = \frac{1}{\sqrt{(10\text{ft})^2 - (6\text{ft})^2}}\frac{3\text{ft}}{\text{s}} = \frac{1}{\sqrt{64\text{ft}^2}}\frac{3\text{ft}}{\text{s}} = \frac{3}{8\text{s}}\,,$$

which is the same as above.

Scores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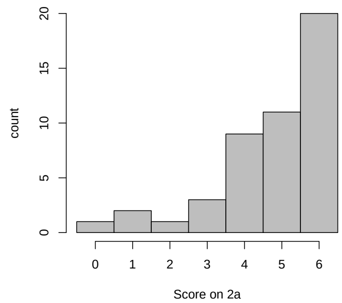
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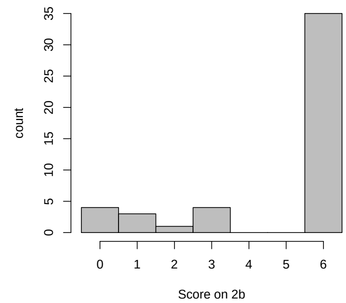
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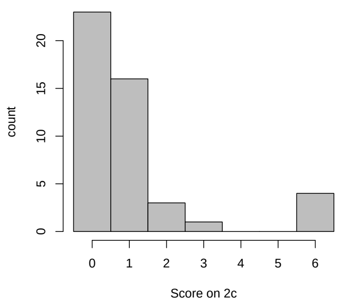
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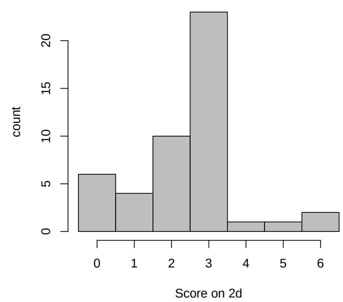
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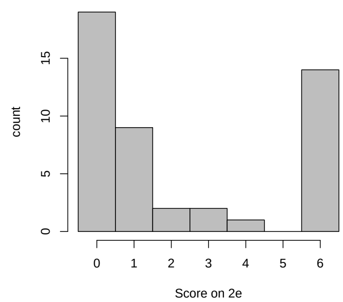
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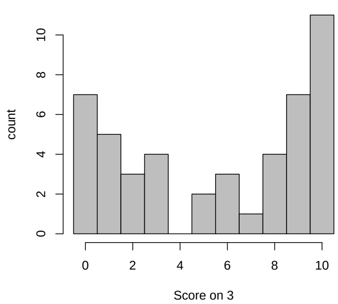
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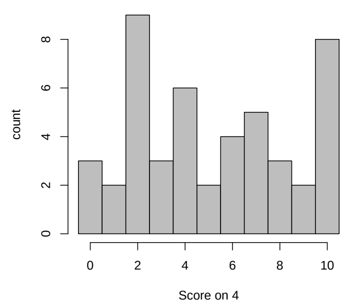
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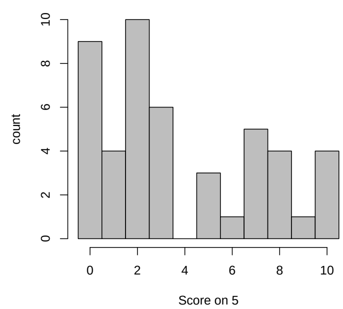
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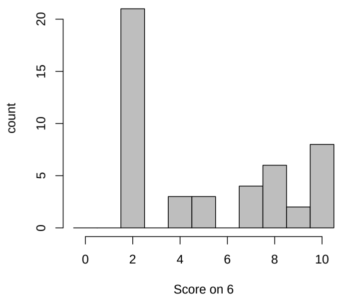
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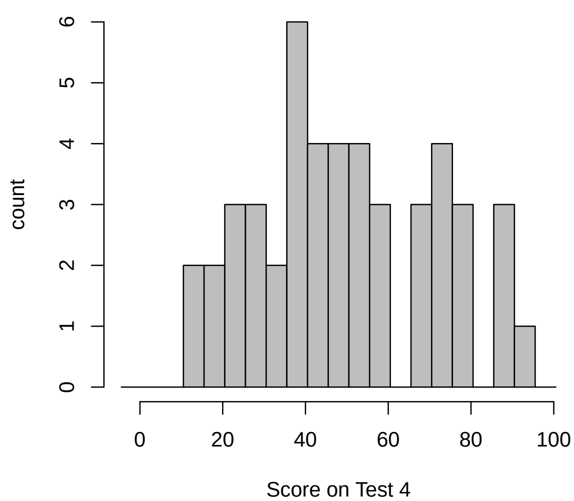
Score on 6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Score on Test 4