score	possible	page
	30	1
	30	2
	20	3
	20	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Determine whether each of the following statements is True or False.
Correct answers are worth +3, incorrect answers are worth −2, and no answer is worth +1.

/3 (a) True False If $\lim_{x\to a} f(x) = f(a)$ then f is differentiable at a.
False. That is the definition of continuous at a point, which is necessary but not sufficient to make f differentiable.

/3 (b) True False If $x \neq 0$ then $\left(\frac{25}{4x^4}\right)\left(\frac{5}{x^3}\right)^{-3} = \frac{2x^5}{5}$.
False. $\left(\frac{25}{4x^4}\right)\left(\frac{5}{x^3}\right)^{-3} = \left(\frac{25}{4x^4}\right)\left(\frac{x^3}{5}\right)^{3} = \left(\frac{25}{4x^4}\right)\left(\frac{x^9}{5^3}\right) = \frac{25x^9}{4x^4\cdot 5^3} = \frac{x^5}{20} \neq \frac{2x^5}{5}$.

/3 (c) True False If a function has a horizontal asymptote, then the graph of the function cannot cross the horizontal asymptote.
False. The horizontal asymptote only tells us about $\lim_{x\to-\infty} f(x)$ or $\lim_{x\to\infty} f(x)$, not what happens in between.

/3 (d) True False The function $f(x) = |x|$ is differentiable on $(-\infty,\infty)$.
False. It is not differentiable at $x = 0$.

/3 (e) True False If $f'(a)$ exists then $\lim_{t\to a} f(t) = f(a)$.
True. If $f'(a)$ exists then f is differentiable at a, which implies f is continuous at a, which by definition means $\lim_{t\to a} f(t) = f(a)$.

/3 (f) True False If f is continuous on $[a,b]$, $f(a) = 1$, and $f(b) = 5$, then there exists $c \in (a,b)$ such that $f(c) = 0$.
False. This is almost the Intermediate Value Theorem but lacks the property that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$.

/3 (g) True False If x is slightly less than 3π then $\sin(x) < 0$.
False. If x is slightly less than 3π then the point $(\cos(x),\sin(x))$ is just above and to the right of $(-1,0)$, so $\sin(x) > 0$.

/3 (h) True False If f is differentiable then $\frac{d}{dx}f(f(f(x))) = f'(f(f(x)))f'(f(x))f'(x)$.
True. This is a double application of the chain rule.

/3 (i) True False $\lim_{x\to\infty} \frac{2+3x^2}{4x^2+5} = \frac{3}{4}$.
True. The highest powers in the numerator and denominator dominate, so

$$\lim_{x\to\infty} \frac{2+3x^2}{4x^2+5} = \lim_{x\to\infty} \frac{3x^2}{4x^2} = \frac{3}{4}.$$

/3 (j) True False $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}+\frac{1}{5}} = \frac{5}{3}$.
False. $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}+\frac{1}{5}} = \frac{\frac{5}{6}}{\frac{7}{10}} = \frac{5}{6}\frac{10}{7} = \frac{25}{21} \neq \frac{5}{3}$.

2. Compute the following derivatives:

/5 (a) $\dfrac{d}{dx}\sqrt{x}\tan(x) =$

$\dfrac{1}{2}x^{-1/2}\tan(x) + \sqrt{x}\sec^2(x)$

/5 (b) $\dfrac{d}{dt}\dfrac{3t}{2+7t^3} =$

$\dfrac{3(2+7t^3) - 3t(7(3t^2))}{(2+7t^3)^2}$

/5 (c) $\dfrac{d}{dx}\left(3x^4 + \cot(x)\right)^9 =$

$9\left(3x^4 + \cot(x)\right)^8 \left(3(4x^3) - \csc^2(x)\right)$

/5 (d) $\dfrac{d}{dx}\sin\left(\sqrt{1+x^4}\right) =$

$\cos\left(\sqrt{1+x^4}\right)\dfrac{1}{2}(1+x^4)^{-1/2}(4x^3)$

/5 (e) $\dfrac{d}{dt}\dfrac{t\cos(t)}{1+4t} =$

$\dfrac{(1\cos(t) + t(-\sin(t)))(1+4t) - t\cos(t)(4)}{(1+4t)^2}$

/5 3. Compute the limit. **Do not use L'Hôpital's rule!**

$\lim_{\theta\to 0}\dfrac{\tan(5\theta)}{\sin(3\theta)} =$

$$\lim_{\theta\to 0}\frac{\frac{\sin(5\theta)}{\cos(5\theta)}}{\sin(3\theta)} = \lim_{\theta\to 0}\frac{\sin(5\theta)}{\cos(5\theta)\sin(3\theta)} = \lim_{\theta\to 0}\frac{1}{\cos(5\theta)}\frac{5}{3}\frac{\sin(5\theta)}{5\theta}\frac{3\theta}{\sin(3\theta)} = \frac{1}{1}\frac{5}{3}(1)(1) = \frac{5}{3}.$$

/10 4. Find $\frac{dy}{dx}$ using implicit differentiation for

$$y^2 \sin(x^2) = x \sin(y^2)\,.$$

Differentiating both sides with respect to x yields

$$2y\frac{dy}{dx}\sin(x^2) + y^2\cos(x^2)2x = \sin(y^2) + x\cos(y^2)2y\frac{dy}{dx}\,.$$

Gathering terms with $\frac{dy}{dx}$ yields

$$2y\frac{dy}{dx}\sin(x^2) - x\cos(y^2)2y\frac{dy}{dx} = \sin(y^2) - y^2\cos(x^2)2x\,.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{\sin(y^2) - y^2\cos(x^2)2x}{2y\sin(x^2) - x\cos(y^2)2y}\,.$$

/10 5. Use differentials (or linear approximation) to estimate the amount of paint (in m^3) needed to apply a coat of paint 0.03 cm thick to a hemispherical dome with diameter 80 m.

The volume of a hemisphere is half that of a sphere, and so is $V = \frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$. Differentiating with respect to r gives $\frac{dV}{dr} = \frac{2}{3}\pi 3r^2$ so $dV = 2\pi r^2 dr$. Inserting the radius of the hemisphere $r = 40\,\mathrm{m}$ and the thickness of the paint $dr = 0.03\,\mathrm{cm}$ yields the paint volume

$$dV = 2\pi(40\,\mathrm{m})^2(0.03\,\mathrm{cm}) = 2\pi\cdot 40^2\cdot 0.03\cdot \mathrm{m}^2\cdot \mathrm{cm} = \frac{2\pi\cdot 40^2\cdot 0.03}{100}\mathrm{m}^3\,.$$

/10 6. If $x^2 + y^2 + z^2 = 9$, $\frac{dx}{dt} = 5$, and $\frac{dy}{dt} = 4$, find $\frac{dz}{dt}$ when $(x, y, z) = (1, 2, 2)$.

Differentiating with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0\,.$$

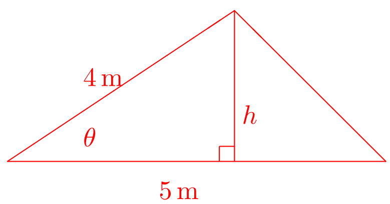
Plugging in the given information yields

$$2 \cdot 1 \cdot 5 + 2 \cdot 2 \cdot 4 + 2 \cdot 2\frac{dz}{dt} = 0\,,$$

so

$$\frac{dz}{dt} = \frac{-(2 \cdot 1 \cdot 5 + 2 \cdot 2 \cdot 4)}{2 \cdot 2} = \frac{-26}{4}\,.$$

/10 7. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.08 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.



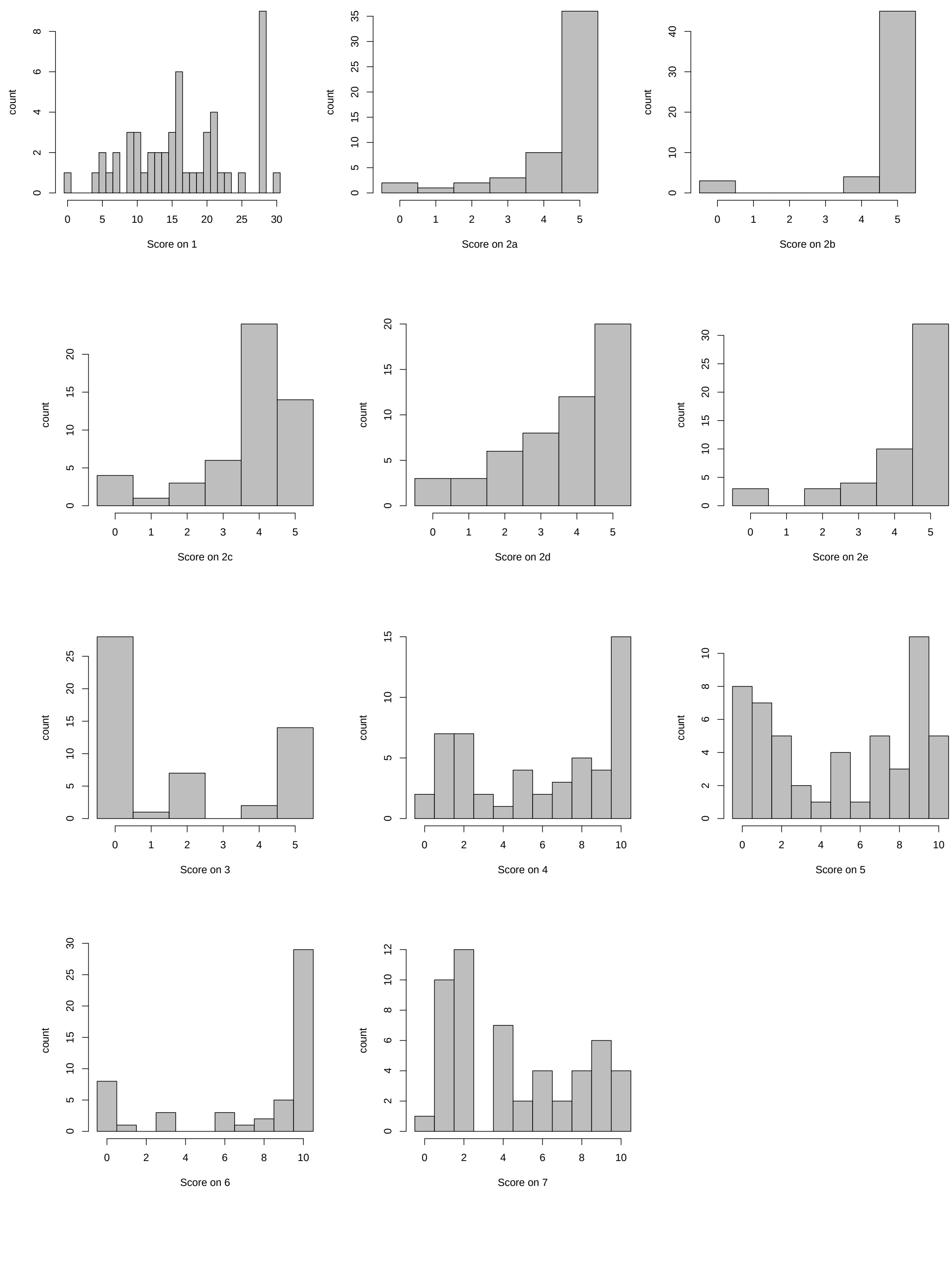
Call the angle θ and make the longer side the base of the triangle, as pictured above. The vertical line segment from the top vertex to the base has length of the height h and hits the base at a right angle. We then have $\sin(\theta) = \frac{h}{4\,\mathrm{m}}$ so $h = 4\,\mathrm{m}\sin(\theta)$. The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(5\,\mathrm{m})(4\,\mathrm{m}\sin(\theta)) = 10\sin(\theta)\mathrm{m}^2\,.$$

Differentiating yields

$$\frac{dA}{dt} = 10\cos(\theta)\mathrm{m}^2\frac{d\theta}{dt} = 10\cos(\pi/3)\mathrm{m}^2 0.08\mathrm{s}^{-1} = 10\frac{1}{2}0.08\frac{\mathrm{m}^2}{\mathrm{s}} = 0.4\frac{\mathrm{m}^2}{\mathrm{s}}\,.$$

Scores


count
Score on 1
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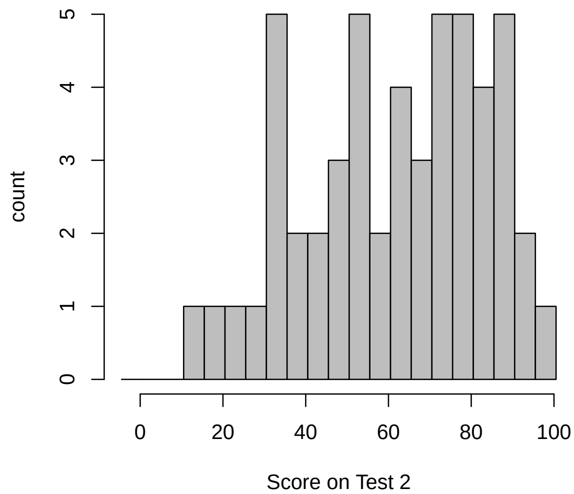
count
0
1
2
3
4
5
0
20
40
60
80
100
Score on Test 2