score	possible	page
	20	1
	30	2
	30	3
	20	4
	100	

Name: ____________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Sketch the graph of a single function f that has all of the following properties:

/4 (a) f has a local maximum at $x = 0$ but is not differentiable there.

/2 (b) $\lim_{x\to 2^+} f(x) = -\infty$.

/2 (c) $\lim_{x\to 2^-} f(x) = \infty$.

/3 (d) f is continuous except at $x = 2$.

/3 (e) f has no inflection points.

/3 (f) $\lim_{x\to +\infty} f(x) = \infty$.

/3 (g) $\lim_{x\to -\infty} f(x) = -2$.

We are only given partial information about the function, and need to deduce more.

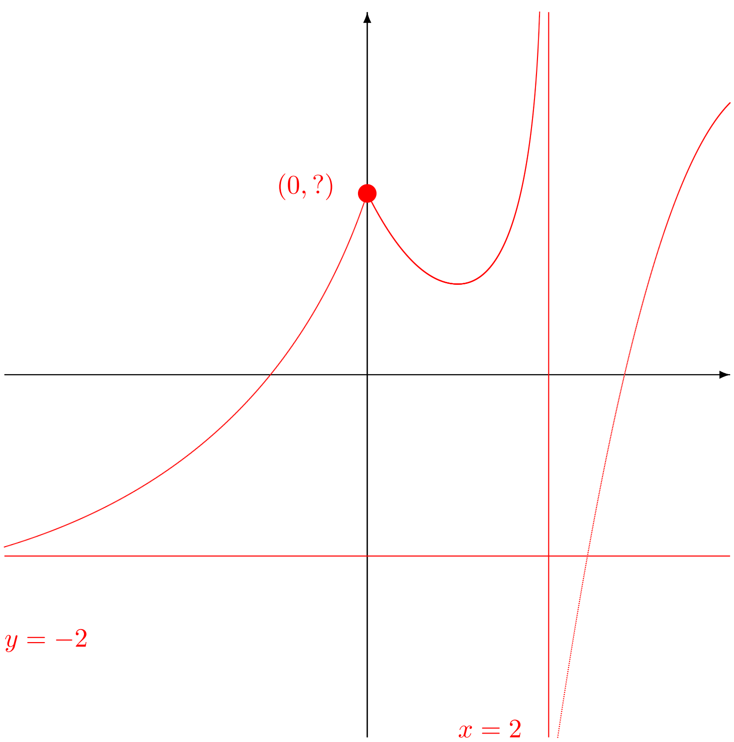
Since $\lim_{x\to 2^+} f(x) = -\infty$, on the interval $(2, ?)$ f must be increasing and concave down. Since it has no inflection points and $\lim_{x\to+\infty} f(x) = \infty$, it must stay increasing and concave down on $(2, \infty)$.

Since $\lim_{x\to 2^-} f(x) = \infty$, on the interval $(?, 2)$ f must be increasing and concave up. Since it has no inflection points and a local max at $x = 0$, on the interval $(0, ?)$ f must be decreasing and concave up and on $(??, 0)$ it must be increasing and concave up. In order to have $\lim_{x\to-\infty} f(x) = -2$ without an inflection point, on $(-\infty, 0)$ it must be increasing and concave up.

Organizing into a chart, we have

f	╯	cusp	╰	→	╯	V.A.	╭
f''	+	DNE	+	+	+	DNE	−
f'	+	DNE	−	0	+	DNE	+
	$(-\infty, 0)$	0	$(0, ?)$	?	$(?, 2)$	2	$(2, \infty)$

(Other features, such as more points where f is not differentiable, are possible but not needed.)



2. Let $f(x) = 2x^3 + 3x^2 - 36x$.

/10 (a) Find the intervals where f is increasing, and the intervals where it is decreasing.

$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$ so the critical numbers are $x = -3$ and $x = 2$. The sign chart is

f	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$
f'	$+$	0	$-$	0	$+$
	$(-\infty, -3)$	-3	$(-3, 2)$	2	$(2, \infty)$

so f is increasing on $(-\infty, -3)$ and $(2, \infty)$ and decreasing on $(-3, 2)$.

/10 (b) Find the intervals where f is concave up, and the intervals where it is concave down.

$f'(x) = 6(2x+1)$ so $f'(x) = 0$ at $x = -1/2$. The sign chart is

f	$\frown$		$\smile$
f''	$-$	0	$+$
	$(-\infty, -1/2)$	$-1/2$	$(-1/2, \infty)$

so f is concave up on $(1/2, \infty)$ and concave down on $(-\infty, -1)$.

/10 (c) Find the absolute maximum and minimum values of f on the interval $[0, 3]$.

The only critical number in the interval is $x = 2$. Evaluating there and at the endpoints we get

$$f(0) = 0\,,$$
$$f(2) = 2(8) + 3(4) - 36(2) = 16 + 12 - 72 = -44\,, \quad \text{and}$$
$$f(3) = 2(27) + 3(9) - 36(3) = 54 + 27 - 108 = -27\,.$$

Thus the absolute maximum is 0 and occurs at $x = 0$ and the absolute minimum is -44 and occurs at $x = 3$.

3. For the function $f(x) = \dfrac{x}{x^2+4}$

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/6 (c) Find the intervals on which f is increasing or decreasing.

/4 (d) Find the local maximum and minimum values of f.

/8 (e) Find the intervals of concavity and the inflection points.

/6 (f) Use the information above to sketch the graph.

(f has odd symmetry, so we could save half the work, but this is optional.)

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is never 0 so there are no vertical asymptotes.

$\lim_{x\to\pm\infty} \frac{x}{x^2+4} = \lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

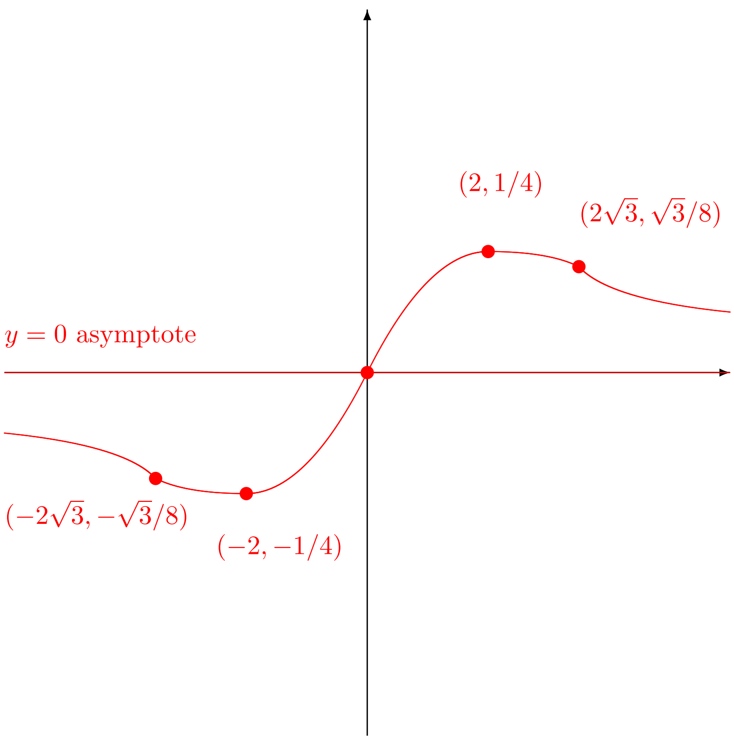
$f'(x) = \frac{1(x^2+4)-x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2} = \frac{(2+x)(2-x)}{(x^2+4)^2}$, which is zero at $x = -2$ and $x = 2$.

$f''(x) = \frac{-2x(x^2+4)^2-(-x^2+4)2(x^2+4)2x}{(x^2+4)^4} = \frac{-2x(x^2+4)-(-x^2+4)4x}{(x^2+4)^3} = \frac{2x(-x^2-4+2x^2-8)}{(x^2+4)^3} = \frac{2x(x^2-12)}{(x^2+4)^3} = \frac{2x(x+\sqrt{12})(x-\sqrt{12})}{(x^2+4)^3}$, which is 0 at $x = 0$, $x = -2\sqrt{3}$, and $x = 2\sqrt{3}$.

Assembling into a chart and checking signs, we have

	$(-\infty,-2\sqrt{3})$	$-2\sqrt{3}$	$(-2\sqrt{3},-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,2\sqrt{3})$	$2\sqrt{3}$	$(2\sqrt{3},\infty)$
f	$\frown$	I.P.	$\smile$	$\to$	$\smile$	I.P.	$\frown$	$\to$	$\frown$	I.P.	$\smile$
f''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$

The is a local max at $x = 2$ with value $f(2) = 2/(2^2+4) = 1/4$ and a local min at $x = -2$ with value $f(-2) = -2/(2^2+4) = -1/4$. There are inflection points at $(-2\sqrt{3}, f(-2\sqrt{3}) = (-2\sqrt{3}, -2\sqrt{3}/(12+4)) = (-2\sqrt{3}, -\sqrt{3}/8)$, $(0, 0)$, and $(2\sqrt{3}, f(2\sqrt{3}) = (2\sqrt{3}, \sqrt{3}/8)$.



/5 4. State the Mean Value Theorem (MVT).

If
- f is continuous on the closed interval $[a, b]$ and
- f is differentiable on the open interval (a, b),

then there exists $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

/5 5. Compute $\lim_{x \to \infty} \frac{2}{x} e^{3x} =$

Plugging in to $\frac{2e^{3x}}{x}$ gives a ∞/∞ indeterminate form, so we can apply L'Hôpital's rule to get $\lim_{x\to\infty} \frac{6e^{3x}}{1} = \infty$.

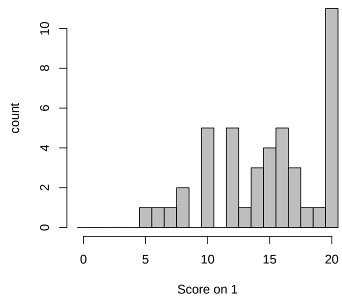
/5 6. Use a linear approximation (or differentials) to estimate $(1.99)^4$.

Set $f(x) = x^4$ so $f'(x) = 4x^3$. Selecting $a = 2$ we have the linear approximation
$$f(x) \approx L_2(x) = f(2) + f'(2)(x - 2) = 16 + 32(x - 2)$$
so $(1.99)^4 = f(1.99) \approx 16 + 32(-0.01) = 16 - 0.32 = 15.68$.

/5 7. For the function $f(x) = 1 + \sqrt{x}$, find the equation for the tangent line at $x = 4$.

We can compute $f'(x) = \frac{1}{2\sqrt{x}}$. The general form of the tangent line at A is $y = f'(A)(x - A) + f(A)$ so we have
$$y = f'(4)(x - 4) + f(4) = \frac{1}{2\sqrt{4}}(x - 4) + (1 + \sqrt{4}) = \frac{1}{4}(x - 4) + 3\,.$$

Scores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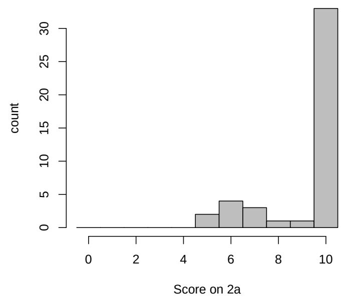
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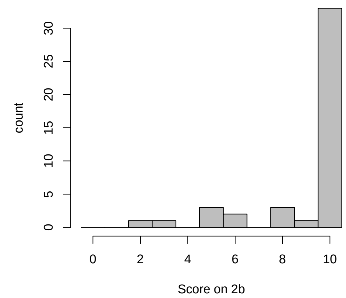
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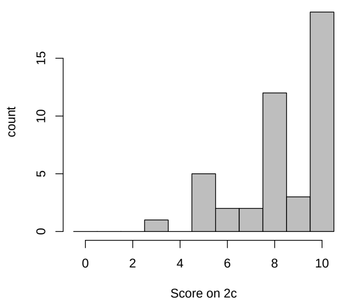
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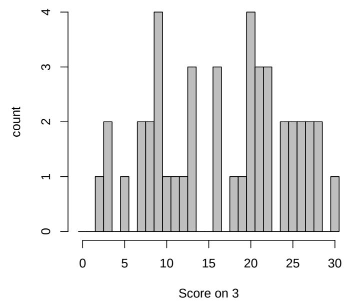
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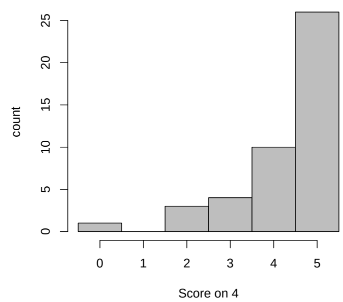
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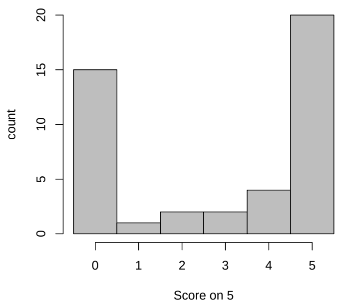
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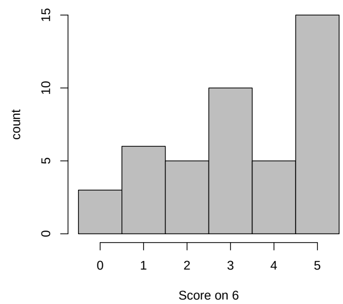
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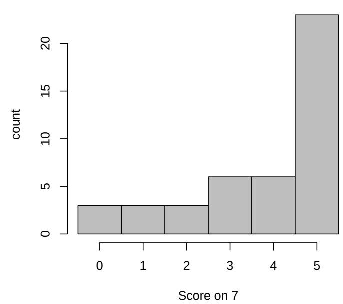
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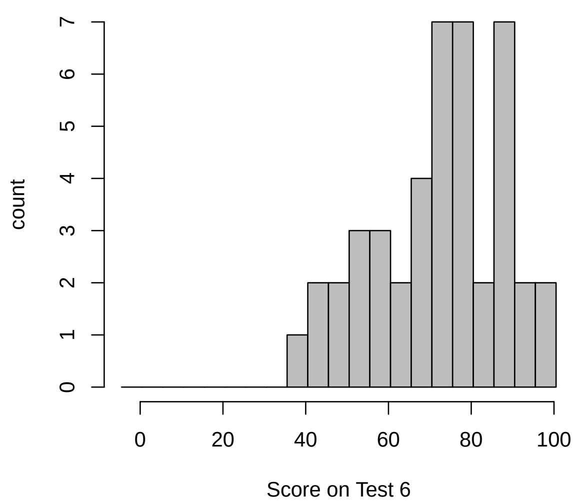
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