score	possible	page
	20	1
	30	2
	25	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/10 1. Complete the definitions:

- Definition: A function f is continuous at a number a if ...

$$\lim_{x\to a} f(x) = f(a)\,.$$

- Definition: A function f is differentiable at a if ...
  $f'(a)$ exists.

Give an example of a function that is one but not the other.

The function $f(x) = |x|$ is continuous (everywhere), but is not differentiable at $a = 0$ since

$$\lim_{h\to 0^-} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \neq \lim_{h\to 0^+} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1\,.$$

/10 2. State the Intermediate Value Theorem. Identify what are its assumptions (hypotheses) and what are its conclusions. Use the Intermediate Value Theorem to show that the equation $5^x = x^2$ has a solution.

If (hypotheses)

- f is continuous on $[a, b]$ and
- $f(a) < N < f(b)$ or $f(a) > N > f(b)$,

then (conclusions) there exists $c \in (a, b)$ such that $f(c) = N$.

Let $f(x) = x^2 - 5^x$, so we want to show a solution to $f(x) = 0$ exists. Since x and 5^x are both continuous, so is $f(x)$. Plugging in, we find

$$f(0) = 0 - 5^0 = -1 < 0 \quad \text{and}$$
$$f(-1) = 1 - 5^{-1} = 4/5 > 0\,.$$

So, by the Intermediate Value Theorem, there must exist $-1 < c < 0$ such that $f(c) = 0$.

3. Compute the following derivatives:

/10 (a) $f(x) = \cot\left(x^2 + \frac{3}{x} + x^{1/4} + \sec(x) - \sin(8)\right)$
$\Rightarrow f'(x) =$

$$-\csc^2\left(x^2 + \frac{3}{x} + x^{1/4} + \sec(x) - \sin(8)\right)\left(2x - 3x^{-2} + \frac{1}{4}x^{-3/4} + \sec(x)\tan(x) + 0\right)$$

/10 (b) $D_x\left[\cos(x)\sin(8 + x^5 + \sin(x))\right] =$

$$-\sin(x)\sin(8 + x^5 + \sin(x)) + \cos(x)\cos(8 + x^5 + \sin(x))(0 + 5x^4 + \cos(x))$$

/10 (c) $y = \sin^5\left(\frac{x}{\cos(x)}\right) \Rightarrow \frac{dy}{dx} =$

$$5\sin^4\left(\frac{x}{\cos(x)}\right)\cos\left(\frac{x}{\cos(x)}\right)\frac{1\cos(x) - x(-\sin(x))}{\cos^2(x)}$$

/15 4. Use implicit differentiation to find an equation for the tangent line to the curve defined by $\sqrt{x+y} = 1 + x^2y^2$ at the point $(0,1)$.

Differentiating both sides with respect to x yields

$$\frac{1}{2\sqrt{x+y}}\left(1+\frac{dy}{dx}\right) = 0 + 2xy^2 + x^2 2y\frac{dy}{dx}\,.$$

Gathering terms with $\frac{dy}{dx}$ yields

$$\frac{1}{2\sqrt{x+y}}\frac{dy}{dx} - 2x^2y\frac{dy}{dx} = -\frac{1}{2\sqrt{x+y}} + 2xy^2\,.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{-\frac{1}{2\sqrt{x+y}} + 2xy^2}{\frac{1}{2\sqrt{x+y}} - 2x^2y} = \frac{-1+4xy^2\sqrt{x+y}}{1-4x^2y\sqrt{x+y}}\,.$$

At $(0,1)$ this yields slope -1 so the tangent line is

$$y = -1(x-0)+1\,.$$

(We should also check that $(0,1)$ is on the curve by plugging in the original equation to get $\sqrt{0+1} = 1+0$.)

/10 5. The radius of a circular disk is given as 25 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate

- the maximum error in the calculated area of the disk, and
- the percentage error in the calculated area of the disk.

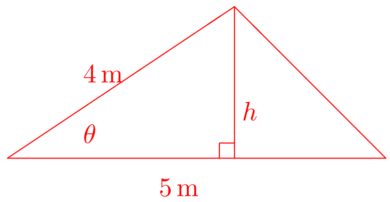
Starting with the area formula $A = \pi r^2$, we can differentiate with respect to r to obtain $\frac{dA}{dr} = 2\pi r$. Converting to differentials yields $dA = 2\pi r\,dr$. The estimated maximum error in the area is then

$$dA = 2\pi(25\,\text{cm})(0.2\,\text{cm}) = 10\pi\,\text{cm}^2\,.$$

The estimated percentage error is

$$\frac{dA}{A}100\% = \frac{2\pi r\,dr}{\pi r^2}100\% = \frac{2\,dr}{r}100\% = \frac{2(0.2\,\text{cm})}{25\,\text{cm}}100\% = 2(0.2)(4)\% = 1.6\%\,.$$

/15 6. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.08 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.



Call the angle θ and make the longer side the base of the triangle, as pictured above. The vertical line segment from the top vertex to the base has length of the height h and hits the base at a right angle. We then have $\sin(\theta) = \frac{h}{4\,\mathrm{m}}$ so $h = 4\,\mathrm{m}\sin(\theta)$. The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(5\,\mathrm{m})(4\,\mathrm{m}\sin(\theta)) = 10\sin(\theta)\mathrm{m}^2\,.$$

Differentiating yields

$$\frac{dA}{dt} = 10\cos(\theta)\mathrm{m}^2\frac{d\theta}{dt} = 10\cos(\pi/3)\mathrm{m}^2 0.08\mathrm{s}^{-1} = 10\frac{1}{2}0.08\frac{\mathrm{m}^2}{\mathrm{s}} = 0.4\frac{\mathrm{m}^2}{\mathrm{s}}\,.$$

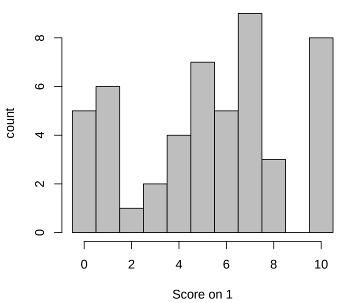
/10 7. Use a linear approximation (or differentials) to estimate $(1.99)^4$.

Set $f(x) = x^4$ so $f'(x) = 4x^3$. Selecting $a = 2$ we have the linear approximation

$$f(x) \approx L_2(x) = f(2) + f'(2)(x-2) = 16 + 32(x-2)$$

so $(1.99)^4 = f(1.99) \approx 16 + 32(-0.01) = 16 - 0.32 = 15.68$.

Scores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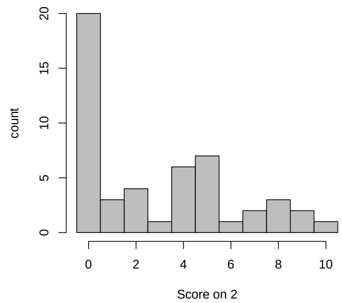
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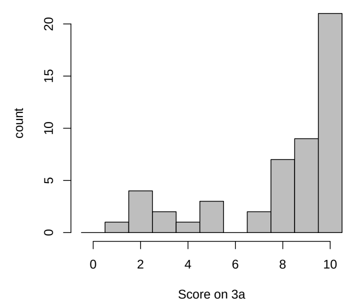
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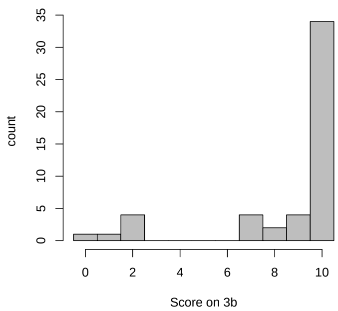
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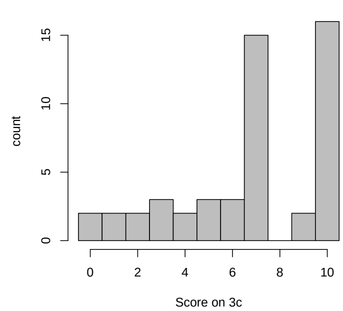
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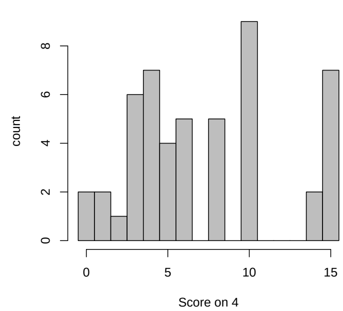
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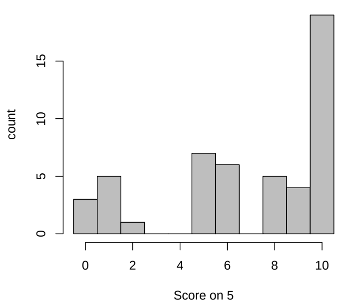
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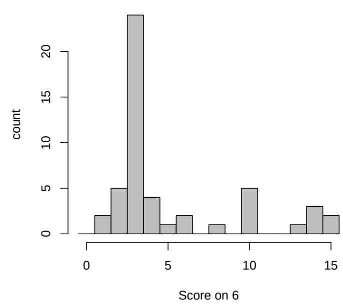
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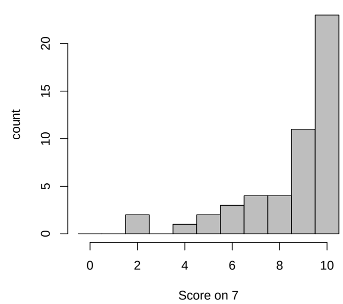
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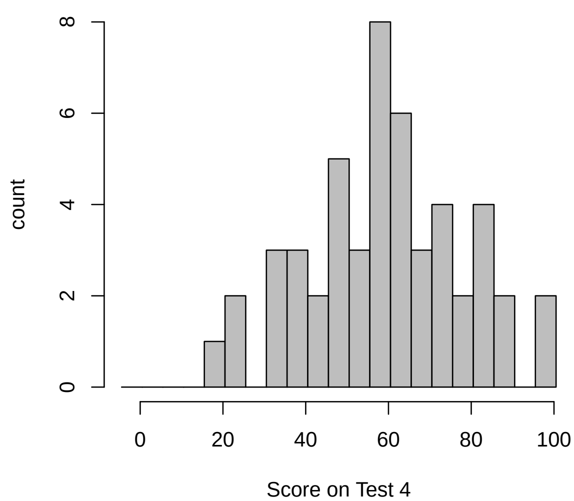
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Score on Test 4