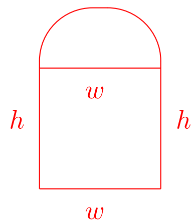
score	possible	page
	20	1
	30	2
	20	3
	30	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/20 1. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30ft, find the dimensions of the window so that the greatest possible amount of light is admitted.



With the diagram above, the perimeter is $2h + w + \pi w/2 = 2h + (1 + \pi/2)w$. Setting equal to 30ft gives the constraint. Solving for h gives $h = 15 - (1 + \pi/2)w/2$.

We want to maximize the area $A = wh + \pi(w/2)^2/2$. Substituting in for h gives

$$A = w\,(15 - (1 + \pi/2)w/2) + \pi(w/2)^2/2 = 15w + (-(1 + \pi/2)/2 + \pi/8)\,w^2 = 15w + (-1 - \pi/4)\,w^2/2\,.$$

Differentiating yields

$$A' = 15 + (-1 - \pi/4)\,w\,,$$

which is 0 when $w = 15/(1 + \pi/4) = 60/(4 + \pi)$. The second derivative is $A'' = -1 - \pi/4 < 0$ so this w is a maximum. Plugging in the constraint, we have

$$h = 15 - \frac{1 + \pi/2}{2}\,\frac{60}{4 + \pi} = \frac{15(4 + \pi) - 30(1 + \pi/2)}{4 + \pi} = \frac{30}{4 + \pi}\,.$$

Therefore the window should be $60/(4 + \pi)$ft wide and the straight part should be $30/(4 + \pi)$ft tall. (Including the circle it is $h + w/2 = 60/(4 + \pi)$ft tall.)

/10 2. Use Newton's method with the initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^5 - x + 1 = 0$. Leave the answer as a fraction.

Setting $f(x) = x^5 - x + 1$ gives $f'(x) = 5x^4 - 1$. Newton's method gives the next approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1^5 - 1 + 1}{5(1^4) - 1} = 1 - \frac{1}{4} = \frac{3}{4}\,.$$

/10 3. Find $f(x)$ if $f''(x) = x^{-2}$ for $x > 0$, $f(1) = 0$, and $f(2) = 0$.

Antidifferentiating once gives $f'(x) = -x^{-1} + C$ and a second time gives $f(x) = -\ln(x) + Cx + D$ with C and D unknown constants. Inserting the given values gives

$$\begin{aligned} 0 &= f(1) = -\ln(1) + C1 + D = C + D \quad \text{and} \\ 0 &= f(2) = -\ln(2) + C2 + D\,. \end{aligned}$$

Subtracting the first equation from the second yields $0 = -\ln(2) + C$ so $C = \ln(2)$. Inserting in the first equation then gives $0 = \ln(2) + D$ so $D = -\ln(2)$. Consequently, $f(x) = -\ln(x) + \ln(2)x - \ln(2)$.

/10 4. Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating intervals and midpoints.

The interval $[a, b] = [0, 4]$ has length 4 and we are using 4 rectangles, so the width of each rectangle is $\Delta x = 1$. The base of the first rectangle is $[0, 1]$, which has midpoint $x_1^* = 1/2$. The second has base $[1, 2]$ and midpoint $x_2^* = 3/2$, the third has base $[2, 3]$ and midpoint $x_3^* = 5/2$, and the fourth has base $[3, 4]$ and midpoint $x_4^* = 7/2$. The area estimate is thus

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x = \sqrt{\frac{1}{2}} \cdot 1 + \sqrt{\frac{3}{2}} \cdot 1 + \sqrt{\frac{5}{2}} \cdot 1 + \sqrt{\frac{7}{2}} \cdot 1\,.$$

/10 5. Compute the following limits and simplify the results.

(a) $\lim_{x\to\infty} \pi/x =$

0

(b) $\lim_{x\to\infty} \cos(\pi/x) =$

$\cos(0) = 1$

(c) $\lim_{x\to\infty} \sin\left(\frac{\pi}{6\cos(\pi/x)}\right) =$

$\sin\left(\frac{\pi}{6(1)}\right) = \frac{1}{2}$

(d) $\lim_{x\to\infty} \log_2\left(\sin\left(\frac{\pi}{6\cos(\pi/x)}\right)\right) =$

$\log_2\left(\frac{1}{2}\right) = -1$

6. Compute the following:

/3 (a) $\frac{d}{dx}[x^x] =$

Set $y = x^x$ so $\ln(y) = x\ln(x)$. Differentiating yields $\frac{y'}{y} = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1$, so $y' = x^x(\ln(x)+1)$.

/3 (b) $\lim_{x\to\infty} x^x = \infty^\infty = \infty$

/4 (c) $\lim_{x\to 0^+} x^x =$

Directly plugging in gives the indeterminate form 0^0. Instead we use properties of exponentials and logarithms and continuity to transform to

$$\lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+} x\ln(x)}.$$

Directly evaluating the inner limit gives indeterminate form $0(-\infty)$. Rewriting to get ∞/∞ form, we can apply L'Hôpital's rule to get

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

The original limit is thus $e^0 = 1$.

7. For the function

$$f(x) = \frac{x}{x^2 - 9}$$

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/6 (c) Find the intervals on which f is increasing or decreasing.

/2 (d) Find the local maximum and minimum values of f.

/6 (e) Find the intervals of concavity and the inflection points.

/10 (f) Use the information above to sketch the graph.

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0, 0)$.

The denominator is 0 and there are vertical asymptotes at $x = -3$ and $x = 3$.

$\lim_{x\to\pm\infty} \frac{x}{x^2-9} = \lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

$f'(x) = \frac{1(x^2-9)-x(2x)}{(x^2-9)^2} = \frac{-x^2-9}{(x^2-9)^2}$, which is undefined at $x = \pm 3$ but is never 0.

$f''(x) = \frac{-2x(x^2-9)^2-(-x^2-9)2(x^2-9)2x}{(x^2-9)^4} = \frac{-2x(x^2-9)-(-x^2-9)4x}{(x^2-9)^3} = \frac{2x(-x^2+9+2x^2+18)}{(x^2-9)^3} = \frac{2x(x^2+27)}{(x^2-9)^3}$, which is undefined at $x = \pm 3$ and 0 at $x = 0$.

Assembling into a chart and checking signs, we have

f		V.A		I.P.		V.A.	
f''	$-$	DNE	$+$	0	$-$	DNE	$+$
f'	$-$	DNE	$-$	$-$	$-$	DNE	$-$
	$(-\infty, -3)$	-3	$(-3, 0)$	0	$(0, 3)$	3	$(3, \infty)$

The are no local maxima or minima. There is an inflection point at $(0, 0)$.