score	possible	page
	20	1
	30	2
	30	3
	20	4
	100	

Name: ____________________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Sketch the graph of a single function that has all of the following properties:

/2 (a) Continuous everywhere.

/2 (b) $f'(x) > 0$ if $|x| < 2$.

/4 (c) $f'(x) < 0$ if $|x| > 2$.

/2 (d) $f'(-2) = 0$.

/2 (e) f is not differentiable at $x = 2$.

/2 (f) $\lim_{x \to 2} |f'(x)| = \infty$.

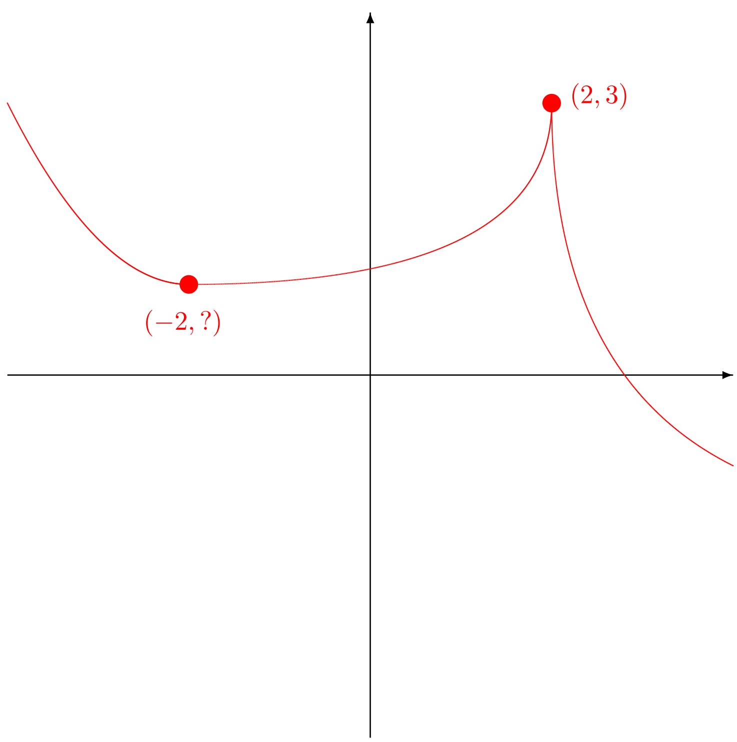
/4 (g) $f''(x) > 0$ if $x \neq 2$.

/2 (h) $f(2) = 3$.

Organizing into a chart, we have

f		$\rightarrow$			
f''	$+$	$+$	$+$	DNE	$+$
f'	$-$	0	$+$	DNE	$-$
	$(-\infty, -2)$	-2	$(-2, 2)$	2	$(2, \infty)$

There is a minimum at $x = -2$ and a cusp at $x = 2$



2. Let $f(x) = 2x^3 - 3x^2 - 12x + 3$

/10 (a) Find the intervals where f is increasing, and the intervals where it is decreasing.

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$ so the critical numbers are $x = 2$ and $x = -1$. The sign chart is

f	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$
f'	$+$	0	$-$	0	$+$
	$(-\infty, -1)$	-1	$(-1, 2)$	2	$(2, \infty)$

so f is increasing on $(-\infty, -1)$ and $(2, \infty)$ and dercreasing on $(-1, 2)$.

/10 (b) Find the intervals where f is concave up, and the intervals where it is concave down.

$f'(x) = 6(2x - 1)$ so $f'(x) = 0$ at $x = 1/2$. The sign chart is

f	$\frown$		$\smile$
f''	$-$	0	$+$
	$(-\infty, 1/2)$	$1/2$	$(1/2, \infty)$

so f is concave up on $(1/2, \infty)$ and concave down on $(-\infty, -1)$.

/10 (c) Find the absolute maximum and minimum values of f on the interval $[-2, 0]$.

The only critical number in the interval is $x = -1$. Evaluating there and at the endpoints we get

$$f(-2) = -16 - 12 + 24 + 3 = -1\,,$$
$$f(-1) = -2 - 3 + 12 + 3 = 10\,, \quad \text{and}$$
$$f(0) = 3\,.$$

Thus the absolute maximum is 10 and occurs at $x = -1$ and the absolute minimum is -1 and occurs at $x = -2$.

3. For the function

$$f(x) = \frac{x}{x^2-9}$$

/2 (a) Find the x- and y-intercepts.

/4 (b) Find any asymptotes.

/6 (c) Find the intervals on which f is increasing or decreasing.

/2 (d) Find the local maximum and minimum values of f.

/6 (e) Find the intervals of concavity and the inflection points.

/10 (f) Use the information above to sketch the graph.

$f(0) = 0$ and no other x makes $f(x) = 0$, so both intercepts are at $(0,0)$.

The denominator is 0 and there are vertical asymptotes at $x = -3$ and $x = 3$.

$\lim_{x\to\pm\infty} \frac{x}{x^2-9} = \lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

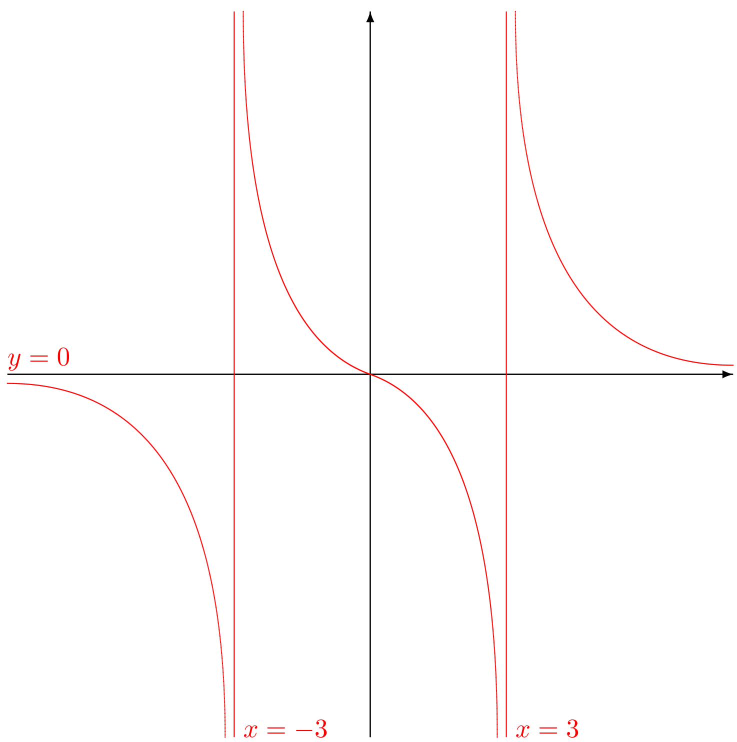
$f'(x) = \frac{1(x^2-9)-x(2x)}{(x^2-9)^2} = \frac{-x^2-9}{(x^2-9)^2}$, which is undefined at $x = \pm 3$ but is never 0.

$f''(x) = \frac{-2x(x^2-9)^2-(-x^2-9)2(x^2-9)2x}{(x^2-9)^4} = \frac{-2x(x^2-9)-(-x^2-9)4x}{(x^2-9)^3} = \frac{2x(-x^2+9+2x^2+18)}{(x^2-9)^3} = \frac{2x(x^2+27)}{(x^2-9)^3}$, which is undefined at $x = \pm 3$ and 0 at $x = 0$.

Assembling into a chart and checking signs, we have

f		V.A		I.P.		V.A.	
f''	$-$	DNE	$+$	0	$-$	DNE	$+$
f'	$-$	DNE	$-$	$-$	$-$	DNE	$-$
	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,3)$	3	$(3,\infty)$

The are no local maxima or minima. There is an inflection point at $(0,0)$.



/5 4. (a) State the Mean Value Theorem (MVT).

If
- f is continuous on the closed interval $[a, b]$ and
- f is differentiable on the open interval (a, b),

then there exists $c \in (a.b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

/5 (b) State why the function

$$f(x) = x^3 - 3x + 2 \quad \text{on the interval} \quad [-2, 2]$$

satisfies each of the hypotheses of the MVT on the given interval. Then find all numbers c that satisfy the conclusion of the MVT.

Since f is a polynomial, it is continuous and differentiable everywhere, and so satisfies the hypotheses.

We want $c \in (-2, 2)$ so that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{(8 - 6 + 2) - (-8 + 6 + 2)}{2 - (-2)} = 1\,.$$

We have $f'(x) = 3x^2 - 3$ so we set up the equation $3c^2 - 3 = 1$, which has solutions $c = \pm\sqrt{4/3}$, both of which are in $(-2, 2)$.

5. Compute the following:

/3 (a) $\dfrac{d}{dx}\,[x^x] =$

Set $y = x^x$ so $\ln(y) = x\ln(x)$. Differentiating yields $\frac{y'}{y} = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1$, so $y' = x^x\,(\ln(x) + 1)$.

/3 (b) $\lim_{x\to\infty} x^x = \infty^\infty = \infty$

/4 (c) $\lim_{x\to 0^+} x^x =$

Directly plugging in gives the indeterminate form 0^0. Instead we use properties of exponentials and logarithms and continuity to transform to

$$\lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+} x\ln(x)}\,.$$

Directly evaluating the inner limit gives indeterminate form $0(-\infty)$. Rewriting to get ∞/∞ form, we can apply L'Hôpital's rule to get

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0\,.$$

The original limit is thus $e^0 = 1$.