score	possible	page
	30	1
	30	2
	20	3
	20	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

1. Compute the following derivatives.

/3 (a) $\frac{d}{dx}\left[x^3\right] = 3x^2$

/3 (b) $\frac{d}{dx}\left[3^x\right] = 3^x \ln(3)$

/3 (c) $\frac{d}{dx}\left[3^3\right] = 0$

/3 (d) $\frac{d}{dx}\left[x^x\right] =$

Set $y = x^x$ so $\ln(y) = x\ln(x)$. Differentiating yields $\frac{y'}{y} = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1$, so $y' = x^x\left(\ln(x)+1\right)$.

/3 (e) $\frac{d}{dx}\left[x^{1/3}\right] = (1/3)x^{-2/3}$

/3 (f) $\frac{d}{dx}\left[x^{-3}\right] = -3x^{-4}$

/3 (g) $\frac{d}{dx}\left[\log_3(x)\right] = \frac{1}{x\ln(3)}$

/3 (h) $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$

/3 (i) $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$

/3 (j) $\frac{d}{dx}\left[\tanh(x)\right] = \text{sech}^2(x)$

2. Compute the following limits. If you use the Squeeze theorem or L'Hôpital's rule, then say so.

/3 (a) $\lim_{x\to 0^+} e^x = e^0 = 1$

/3 (b) $\lim_{x\to\infty} e^x = \infty$

/3 (c) $\lim_{x\to 0^+} \ln(x) = -\infty$

/3 (d) $\lim_{x\to\infty} \ln(x) = \infty$

/3 (e) $\lim_{x\to 0^+} \frac{2}{x}\sin(3x) =$

$\lim_{x\to 0^+} 6\frac{\sin(3x)}{3x} = \lim_{t\to 0^+} 6\frac{\sin(t)}{t} = 6$ if we remember the last limit. Alternatively, if we note that plugging in to $\frac{2\sin(3x)}{x}$ gives a $0/0$ indeterminate form, we can apply L'Hôpital's rule to get $\lim_{x\to 0^+} \frac{6\cos(3x)}{1} = 6$.

/3 (f) $\lim_{x\to\infty} \frac{2}{x}\sin(3x) =$

Noting that $-\frac{2}{x} \le \frac{2}{x}\sin(3x) \le \frac{2}{x}$ for $x > 0$ and $\lim_{x\to\infty} -\frac{2}{x} = \lim_{x\to\infty}\frac{2}{x} = 0$, we apply the squeeze theorem to conclude the limit is 0.

/3 (g) $\lim_{x\to 0^+} \frac{2}{x}e^{3x} = 2e^0 \lim_{x\to 0^+}\frac{1}{x} = 2\infty = \infty$

/3 (h) $\lim_{x\to\infty} \frac{2}{x}e^{3x} =$

Plugging in to $\frac{2e^{3x}}{x}$ gives a ∞/∞ indeterminate form, so we can apply L'Hôpital's rule to get $\lim_{x\to\infty}\frac{6e^{3x}}{1} = \infty$

/3 (i) $\lim_{x\to -1^+} \sin^{-1}(x) = -\pi/2$

/3 (j) $\lim_{x\to\infty} \sinh(x) = \infty$

/10 3. If an electrostatic field E acts on a liquid polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{\cosh(E)}{\sinh(E)} - \frac{1}{E}\,.$$

Show that $\lim_{E\to 0^+} P(E) = 0$.

Plugging in gives a $\infty - \infty$ indeterminate form. Combining fractions gives

$$\lim_{E\to 0^+} \frac{E\cosh(E) - \sinh(E)}{E\sinh(E)}$$

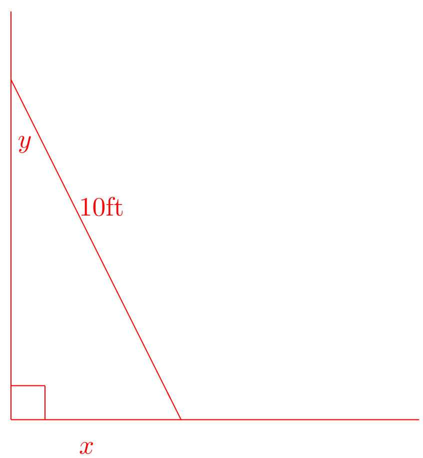
which gives 0/0 indeterminate form. Applying L'Hôpital's rule yields

$$\lim_{E\to 0^+} \frac{\cosh(E) + E\sinh(E) - \cosh(E)}{\sinh(E) + E\cosh(E)} = \lim_{E\to 0^+} \frac{E\sinh(E)}{\sinh(E) + E\cosh(E)}$$

which still has 0/0 indeterminate form. Applying L'Hôpital's rule again yields

$$\lim_{E\to 0^+} \frac{\sinh(E) + E\cosh(E)}{\cosh(E) + \cosh(E) + E\sinh(E)} = \frac{0 + 0\cdot 1}{1 + 1 + 0\cdot 0} = 0\,.$$

/10 4. A ladder 10 ft long is leaning against a vertical wall. It starts slipping, such that the bottom of the ladder slides away from the base of the wall at a speed of 2ft/s. Draw and label a diagram illustrating this scenario. How fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?



We know $\frac{dx}{dt} = 2\text{ft/s}$ and want to know $\frac{dy}{dt}$. We can relate x and y by $y = \arcsin(x/(10\text{ft}))$. Differentiating yields

$$\frac{dy}{dt} = \frac{1}{\sqrt{1 - ((x/(10\text{ft}))^2}}\frac{1}{10\text{ft}}\frac{dx}{dt}\,.$$

Inserting $\frac{dx}{dt} = 2\text{ft/s}$ and $x = 6\text{ft}$ yields

$$\frac{dy}{dt} = \frac{1}{\sqrt{1 - (3/5)^2}}\frac{1}{5\text{s}} = \frac{1}{\sqrt{16/25}}\frac{1}{5\text{s}} = \frac{1}{4\text{s}}\,.$$

Alternatively, you could use $\sin(y) = x/(10\text{ft})$ and implicitly differentiate to get $\cos(y)\frac{dy}{dt} = \frac{1}{10\text{ft}}\frac{dx}{dt}$ and note $\cos(y) = \sqrt{(10\text{ft})^2 - x^2}/(10\text{ft})$. We then have

$$\frac{dy}{dt} = \frac{10\text{ft}}{\sqrt{(10\text{ft})^2 - x^2}}\frac{1}{10\text{ft}}\frac{dx}{dt}\,,$$

which is the same as above.

/10 5. State

- the definition of "Continuous" and
- the definition of "Differentiable".

Give an example of a function that is one but not the other.

- A function f is continuous at a if
$$\lim_{x\to a} f(x) = f(a)\,.$$
- A function f is differentiable at a if $f'(a)$ exists.

The function $f(x) = |x|$ is continuous (everywhere), but is not differentiable at $a = 0$ since

$$\lim_{h\to 0^-} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \neq \lim_{h\to 0^+} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1\,.$$

/10 6. Use a linear approximation (or differentials) to estimate $(8.03)^{2/3}$.

Set $f(x) = x^{2/3}$ so $f'(x) = (2/3)x^{-1/3}$. Selecting $a = 8$ we have the linear approximation

$$f(x) \approx L_8(x) = f(8) + f'(8)(x-8) = 4 + \frac{1}{3}(x-8)$$

so $(8.03)^{2/3} = f(8.03) \approx 4 + \frac{0.03}{3} = 4.01$.