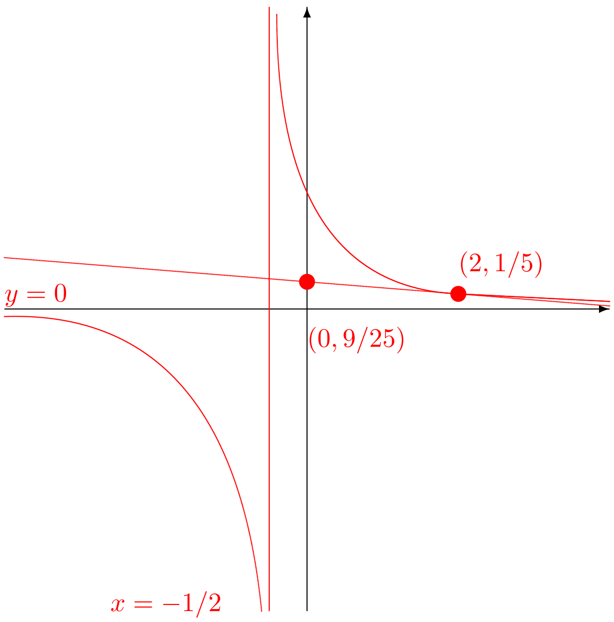
score	possible	page
	20	1
	30	2
	25	3
	25	4
	100	

Name: ______________________________

Show your work!
You may not give or receive any assistance during a test, including but not limited to using notes, phones, calculators, computers, or another student's solutions. (You may ask me questions.)

/10 1. For the function $f(x) = (2x+1)^{-1}$, the tangent line at $x = 2$ has equation $y = (-2/25)(x-2) + 1/5$. Graph $f(x)$ and the tangent line.

There is a vertical asymptote at $x = -1/2$ and a horizontal asymptote at $y = 0$. Near the vertical asymptote, $\lim_{x\to(-1/2)^+} f(x) = \infty$ and $\lim_{x\to(-1/2)^-} f(x) = -\infty$. The tangent line has y-intercept at $9/25$.



/10 2. State

- the definition of "Continuous" and
- the definition of "Differentiable".

Give an example of a function that is one but not the other.

- A function f is continuous at a if
$$\lim_{x\to a} f(x) = f(a)\,.$$
- A function f is differentiable at a if $f'(a)$ exists.

The function $f(x) = |x|$ is continuous (everywhere), but is not differentiable at $a = 0$ since

$$\lim_{h\to 0^-} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \neq \lim_{h\to 0^+} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1\,.$$

3. Compute the following derivatives:

/10 (a) $f(x) = \sqrt{x^2 + \dfrac{3}{x} + x^{3/4} + \cot(x) - \sin(7)}$
$\Rightarrow f'(x) =$

$$\frac{2x - 3x^{-2} + \frac{3}{4}x^{-1/4} - \csc^2(x) - 0}{2\sqrt{x^2 + \frac{3}{x} + x^{3/4} + \cot(x) - \sin(7)}}$$

/10 (b) $D_x \left[\cos(x)\sin(8 + x^5 + 3x)\right] =$

$$-\sin(x)\sin(8 + x^5 + 3x) + \cos(x)\cos(8 + x^5 + 3x)(0 + 5x^4 + 3)$$

/10 (c) $y = \tan^5\left(6x^3 - 7x\right) \Rightarrow \dfrac{dy}{dx} =$

$$5\tan^4\left(6x^3 - 7x\right)\sec^2\left(6x^3 - 7x\right)\left(18x^2 - 7\right)$$

/15 4. Use implicit differentiation to find an equation for the tangent line to the curve defined by $y^3 + x^2y^4 = 1 + 2x$ at the point $(0, 1)$.

Differentiating both sides with respect to x yields

$$3y^2\frac{dy}{dx} + 2xy^4 + 4x^2y^3\frac{dy}{dx} = 0 + 2\,.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}\,.$$

At $(0, 1)$ this yeilds slope $2/3$ and so the tangent line is

$$y = \frac{2}{3}(x - 0) + 1\,.$$

(We should also check that $(0, 1)$ is on the curve by plugging in the original equation to get $1{+}0 = 1{+}0$.)

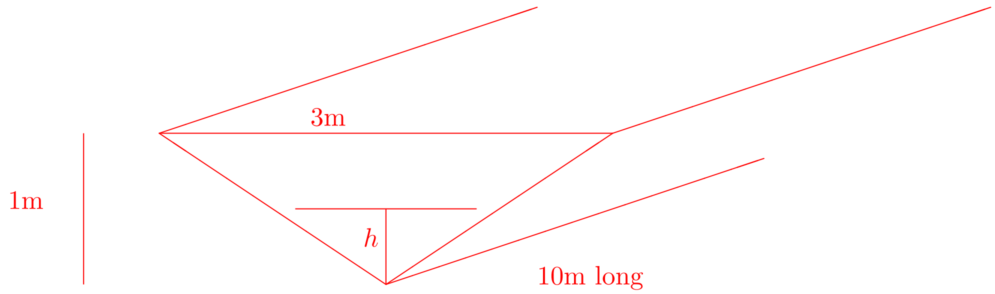
/10 5. State the Intermediate Value Theorem. Identify what are its assumptions (hypotheses) and what are its conclusions.

If (hypotheses)

- f is continuous on $[a, b]$ and
- $f(a) < N < f(b)$ or $f(a) > N > f(b)$,

then (conclusions) there exists $c \in (a, b)$ such that $f(c) = N$.

/15 6. A trough is 10m long and its ends have the shape of isosceles triangles that are 3m across at the top and have a height of 1m. The trough is being filled with water at a rate of $12\text{m}^3/\text{min}$. Draw and label a diagram illustrating this scenario. How fast is the water level rising when it is 0.5m deep?



Since the triangular end has proportion $3/1$, for a given height of water h the width is $3h$, so the area of the end is $\frac{3}{2}h^2$ and the volume of water is $v = 15h^2\text{m}$. We are given $\frac{dv}{dt} = 12\text{m}^3/\text{min}$ and want $\frac{dh}{dt}$ when $h = 0.5\text{m}$. Differentiating with respect to t gives $\frac{dv}{dt} = 30h\frac{dh}{dt}\text{m}$, so

$$\frac{dh}{dt} = \frac{1}{30h\text{m}}\frac{dv}{dt} = \frac{1}{30/2\text{m}^2}12\text{m}^3/\text{min} = \frac{4}{5}\text{m}/\text{min}\,.$$

/10 7. Use a linear approximation (or differentials) to estimate $(8.03)^{2/3}$.

Set $f(x) = x^{2/3}$ so $f'(x) = (2/3)x^{-1/3}$. Selecting $a = 8$ we have the linear approximation

$$f(x) \approx L_8(x) = f(8) + f'(8)(x-8) = 4 + \frac{1}{3}(x-8)$$

so $(8.03)^{2/3} = f(8.03) \approx 4 + \frac{0.03}{3} = 4.01$.