The test is Friday April 5. Here are some sample questions, some of which are from old tests. No books or notes are allowed.

1. Conditioning:

   (a) State the formal definition for the relative condition number of a problem and explain what all the terms are.

   (b) Prove: Let $A \in \mathbb{C}^{m\times m}$ be nonsingular and consider the equation $Ax = b$. The relative condition number is $\kappa(A) = \|A\|\|A^{-1}\|$ for the problem of computing

   - b given x with respect to perturbations of x.
   - x given b with respect to perturbations of b.
   - x given b with respect to perturbations of A.

2. Stability:

   (a) State the formal and informal definitions of stability.

   (b) State the formal and informal definitions of backwards stability.

   (c) Prove:
   Suppose a backward stable algorithm is applied to solve a problem $f : X \to Y$ with condition number κ on a computer satisfying the usual axioms of floating point arithmetic. Then the relative errors satisfy

   $$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\epsilon_{\text{machine}})\,.$$

   (d) To solve the nonlinear system of equations

   $$x_1 + ax_2^2 + b = 0 \quad \text{and}$$
   $$cx_2 + d = 0$$

   for x_1 and x_2, an algorithm is proposed that first computes x_2 by $x_2 = -d/c$ and then substitutes the result in to compute x_1 by $x_1 = -ax_2^2 - b$. Show that this algorithm is backward stable or demonstrate that it is not.

3. Least-Squares:

   (a) Let $b \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$ of full rank be fixed. Let $x \in \mathbb{C}^n$ be the minimizer of $\|b - Ax\|_2$ and $y = Ax$. Determine and prove the condition number of

   - y with respect to perturbations of b.
   - y with respect to perturbations of A.
   - x with respect to perturbations of b.

   (b) Which algorithms for least squares are stable, and which are unstable?