AN INTRODUCTION TO EVERYWHERE CONTINUOUS, NOWHERE DIFFERENTIABLE FUNCTIONS

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APRIL 23, 2013

Abstract. In calculus courses, students learn the properties of continuous and differentiable functions. One extremely important fact about differentiable functions is that they are continuous. Students are also taught that the converse is not true, which can be surprising. Even more surprising is the fact that a function can be continuous everywhere, but differentiable nowhere. We explore the properties of these types of functions; specifically, we introduce the notion of an everywhere continuous, nowhere differentiable function, using the famed Weierstrass function as the prime example. We then examine the Weierstrass function in more detail.

1. Introduction

Students of elementary differential calculus are taught a very important fact about functions of one-real variable early in their studies: *If a function is differentiable, then it is continuous.* The converse is not necessarily true, i.e. functions can be continuous but not differentiable. Differentiable functions are the main object of study in a normal differential calculus course. Formally, a real-valued function f is *differentiable* at a point $x \in (a, b) \subseteq \mathbb{R}$ if

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

exists. The function $f'(x)$ is called the *derivative of f at x* [1]. Another way the derivative is explained is that it is the slope of a curve at a given point. A real-valued function f is *continuous* at a point $c \in [a, b] \subseteq \mathbb{R}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all points $x \in [a, b]$ for which $|x - c| < \delta$ [1]. Continuous functions can be intuited as functions which have no holes or breaks where they are defined. Normally, the functions presented in calculus courses will be rather nice, i.e. having continuity and differentiability properties that are easy to identify. The differentiability and continuity of a function are intimately related by the following theorem discussed in calculus courses.

Theorem 1 ([1]). *If a function f is differentiable at a point $c \in \mathbb{R}$, then f is continuous at c.*

The notion in Theorem 1 can be extended to an interval $[a, b]$ over the set of real numbers $\mathbb{R}$. A function f is called continuous over $[a, b]$ if f is continuous at every point in $[a, b]$. Similarly, if a function f is differentiable over $[a, b]$, then f is continuous over $[a, b]$. At some point in the 19^{th} century, mathematicians wondered if the converse to the previous statement was true, i.e. if a function is continuous, is it differentiable? This is not the case, as evidenced by the function $f(x) = |x|$ and its derivative, which are shown below.

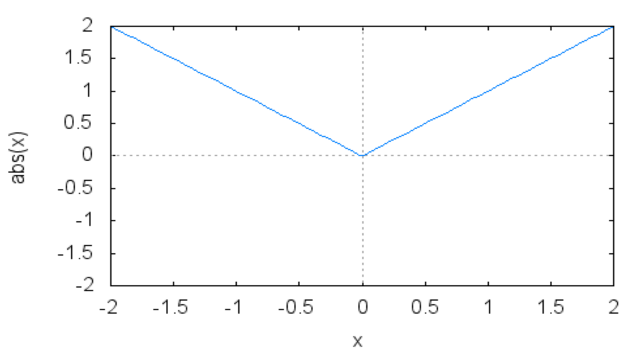

Figure 1.1: The absolute value function over $[-2, 2]$.

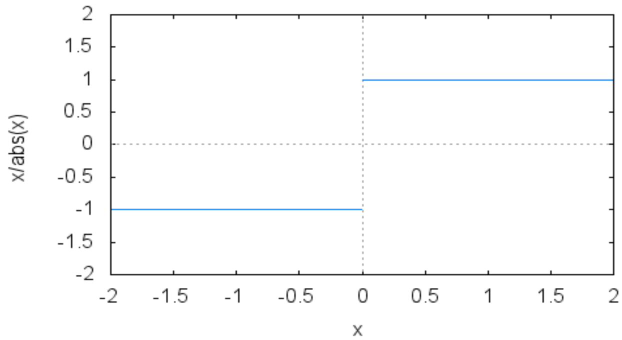

Figure 1.2: The derivative of the absolute value function over $[-2, 2]$.

These mathematicians asked if everywhere continuous functions (i.e. continuous everywhere on the real line) could be nowhere differentiable (i.e. differentiable nowhere on the real line). This happens to be the case, and in 1872, Karl Weierstrass presented his famed Weierstrass function to the Royal Academy of Science in Berlin, Germany. The graph to this function is shown in Figure 1.3.

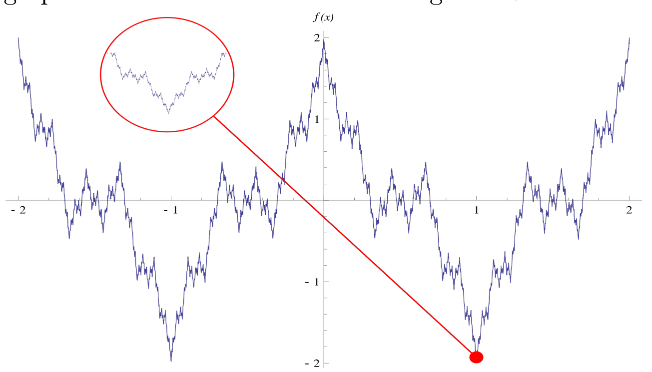

Figure 1.3: The Weierstrass function over $[-2, 2]$ [5].

The Weierstrass function, as it is presented in Figure 1.3, is defined by the following theorem.

Theorem 2 ([3])**.** *Consider the following function:*

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

for $0 < a < 1$ *and* b *a positive odd integer greater than 1 such that* $ab > 1 + \frac{3}{2}\pi$*. This function is everywhere continuous but nowhere differentiable.*

The proof that this function is truly everywhere continuous and nowhere differentiable uses few notions from higher mathematics courses, so undergraduates should not have trouble understanding the content presented. The proof itself will be split into two sections: the first examines nowhere differentiability and the second examines everywhere continuity. The topic of the Weierstrass function, and everywhere continuous, nowhere differentiable functions in general, presents an extremely interesting counterexample to the converse of one of the most important theorems in differential calculus.

2. Nowhere Differentiability

When we prove that the Weierstrass function is nowhere differentiable, we will examine the left and right-hand difference quotients of the Weierstrass function and show a discrepancy in their signs. Specifically, we will consider an arbitrary point $x_0 \in \mathbb{R}$ and construct two points $y_m, z_m \in \mathbb{R}$, which are to the left and right of and converge to x_0, using strict assumptions. We will derive a simple formula for each difference quotient using our constructed points and show a discrepancy between them. Then, we will show that the derivative cannot exist from either side because they approach infinity. Furthermore, they will have different signs, which means that the derivative at any point will always oscillate.

2.1. [3] **The Weierstrass Function is Nowhere Differentiable.**

Proof. We will prove the proposition by comparing the left and right-hand difference quotients, and achieving a contradiction. We start by letting x_0 be a fixed, arbitrary real number. We choose α_m an integer where $m \in \mathbb{N}$ is arbitrary such that $b^m x_0 - \alpha_m \in (-\frac{1}{2}, \frac{1}{2}]$ and we define $x_{m+1} = b^m x_0 - \alpha_m$. If we put $y_m = \frac{\alpha_m - 1}{b^m}$ and $z_m = \frac{\alpha_m + 1}{b^m}$, then

$$y_m - x_0 = -\frac{1 + x_{m+1}}{b^m} < 0 < \frac{1 - x_{m+1}}{b^m} = z_m - x_0$$

and therefore $y_m < x_0 < z_m$. It follows that

$$\begin{aligned}\lim_{m\to\infty}(y_m - x_0) &= \lim_{m\to\infty} -\frac{1 + x_{m+1}}{b^m} \\ &= 0,\end{aligned}$$

implying that

$$\lim_{m\to\infty} y_m = x_0.$$

Note that this justification is the same for z_m, so we see that as $m \to \infty$, $y_m \to x_0$ from the left and $z_m \to x_0$ from the right. We will first look at the left-hand difference quotient,

$$\begin{aligned}\frac{f(y_m) - f(x_0)}{y_m - x_0} &= \frac{\sum_{n=0}^{\infty} a^n \cos(b^n \pi y_m) - \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_0)}{y_m - x_0} \\ &= \sum_{n=0}^{\infty} a^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{y_m - x_0} \\ &= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} \\ &+ \sum_{n=0}^{\infty} a^{m+n} \frac{\cos(b^{m+n} \pi y_m) - \cos(b^{m+n} \pi x_0)}{y_m - x_0} \\ &= S_1 + S_2,\end{aligned}$$

where S_1 and S_2 are partial sums. We will consider each of these partial sums separately, by first considering S_1.Well, since

$$\begin{aligned}S_1 &= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} \\ &= \sum_{n=0}^{m-1} (-\pi)(ab)^n \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{b^n \pi (y_m - x_0)}{2}}\end{aligned}$$

by the trigonometric identity $\cos(\theta_1) - \cos(\theta_2) = -2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right)$, and since

$$\left|\frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{b^n \pi (y_m - x_0)}{2}}\right| \leq 1$$

because $|\frac{\sin x}{x}| \leq 1$, we get

$$\begin{aligned}|S_1| &= \left|\sum_{n=0}^{m-1} (-\pi)(ab)^n \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{b^n \pi (y_m - x_0)}{2}}\right| \\ &\leq \sum_{n=0}^{m-1} \pi (ab)^n = \frac{\pi[(ab)^m - 1]}{ab - 1} \leq \frac{\pi (ab)^m}{ab - 1}.\end{aligned}$$

Hence, there exists an $\epsilon_1 \in [-1, 1]$ such that

$$S_1 = \epsilon_1 \frac{\pi(ab)^m}{ab-1}. \tag{2.1}$$

If we consider the second partial sum S_2, and look at $\cos(b^{m+n}\pi y_m)$, we get, because we chose b to be an odd integer and $\alpha_m \in \mathbb{Z}$

$$\begin{aligned}\cos(b^{m+n}\pi y_m) &= \cos(b^{m+n}\pi \frac{\alpha_m - 1}{b^m}) \\ &= \cos(b^n \pi(\alpha_m - 1)) \\ &= [(-1)^{b^n}]^{\alpha_m - 1} \\ &= -(-1)^{\alpha_m}.\end{aligned}$$

We now consider the second trignometric part of S_2, $\cos(b^{m+n}\pi x_0)$, and we get

$$\begin{aligned}\cos(b^{m+n}\pi x_0) &= \cos\left(b^{m+n}\pi \frac{\alpha_m + x_{m+1}}{b^m}\right) \\ &= \cos(b^n \pi \alpha_m)\cos(b^n \pi x_{m+1}) - \sin(b^n \pi \alpha_m)\sin(b^n \pi x_{m+1}) \\ &= [(-1)^{b^n}]^{\alpha_m}\cos(b^n \pi x_{m+1}) - 0 \\ &= (-1)^{\alpha_m}\cos(b^n \pi x_{m+1}).\end{aligned}$$

This means we can express S_2 as

$$\begin{aligned}S_2 &= \sum_{n=0}^{\infty} a^{m+n} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n \pi x_{m+1})}{-\frac{1+x_{m+1}}{b^m}} \\ &= (ab)^m(-1)^{\alpha_m} \sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_{m+1})}{1+x_{m+1}}.\end{aligned}$$

By the assumption that $a \in (0, 1)$, each term in the series

$$\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_{m+1})}{1+x_{m+1}}$$

is non-negative and, because $x_{m+1} \in (-\frac{1}{2}, \frac{1}{2}]$, we can find a lower bound of this series,

$$\begin{aligned}\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_{m+1})}{1+x_{m+1}} &\geq \frac{1+\cos(\pi x_{m+1})}{1+x_{m+1}} \\ &\geq \frac{1}{1+\frac{1}{2}} \\ &= \frac{2}{3}.\end{aligned}$$

Hence, there exists an $\eta_1 > 1$ such that

$$S_2 = (ab)^m(-1)^{\alpha_m}\eta_1 \frac{2}{3}. \tag{2.2}$$

By considering (2.1) and (2.2), we see that

$$\begin{aligned}\frac{f(y_m) - f(x_0)}{y_m - x_0} &= (ab)^m(-1)^{\alpha_m}\eta_1 \frac{2}{3} + \epsilon_1 \frac{\pi(ab)^m}{ab-1} \\ &= (-1)^{\alpha_m}(ab)^m \eta_1 \left(\frac{2}{3} + \frac{\epsilon_1}{\eta_1} \cdot \frac{\pi}{ab-1}\right),\end{aligned} \tag{2.3}$$

where the sign of $\frac{\epsilon_1}{\eta_1}$ depends on $(-1)^{\alpha_m}$.

If we now consider the right-hand difference quotient $\frac{f(z_m)-f(x_0)}{z_m-x_0}$, the process is much the same. We express the difference quotient as two partial sums, shown as

$$\frac{f(z_m)-f(x_0)}{z_m-x_0} = S_1' + S_2'.$$

Similar to what we showed in (2.1), we can show that there exists an $\epsilon_2 \in [-1,1]$ such that

$$S_1' = \epsilon_2 \frac{\pi(ab)^m}{ab-1}. \tag{2.4}$$

Considering the cosine terms in S_2', we arrive at, because b is odd and $\alpha_m \in \mathbb{Z}$,

$$\begin{aligned} \cos(b^{m+n}\pi z_m) &= \cos\left(b^{m+n}\pi\frac{\alpha_m+1}{b^m}\right) \\ &= \cos(b^n\pi(\alpha_m+1)) \\ &= [(-1)^{b^n}]^{\alpha_m+1} \\ &= -(-1)^{\alpha_m}. \end{aligned}$$

This means that

$$\begin{aligned} S_2' &= \sum_{n=0}^{\infty} a^{m+n}\frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n\pi x_{m+1})}{\frac{1-x_{m+1}}{b^m}} \\ &= -(ab)^m(-1)^{\alpha_m}\sum_{n=0}^{\infty} a^n\frac{1+\cos(b^n\pi x_{m+1})}{1-x_{m+1}}. \end{aligned}$$

Thus, we can find a lower bound for the series

$$\sum_{n=0}^{\infty} a^n\frac{1+\cos(b^n\pi x_{m+1})}{1-x_{m+1}}$$

by

$$\begin{aligned} \sum_{n=0}^{\infty} a^n\frac{1+\cos(b^n\pi x_{m+1})}{1-x_{m+1}} &\geq \frac{1+\cos(\pi x_{m+1})}{1-x_{m+1}} \\ &\geq \frac{1}{1-(-\frac{1}{2})} \\ &= \frac{2}{3}. \end{aligned}$$

Hence, as before, there exists an $\eta_2 > 1$ such that

$$S_2' = -(-1)^{\alpha_m}(ab)^m\eta_2\frac{2}{3}. \tag{2.5}$$

Thus, by (2.4) and (2.5), we see that

$$\begin{aligned} \frac{f(z_m)-f(x_0)}{z_m-x_0} &= -(-1)^{\alpha_m}(ab)^m\eta_2\frac{2}{3} + \epsilon_2\frac{\pi(ab)^m}{ab-1} \\ &= -(-1)^{\alpha_m}(ab)^m\eta_2\left(\frac{2}{3}+\frac{\epsilon_2}{\eta_2}\cdot\frac{\pi}{ab-1}\right), \end{aligned} \tag{2.6}$$

where the sign of $\frac{\epsilon_2}{\eta_2}$ depends on $-(-1)^{\alpha_m}$.

We assumed that $ab > 1+\pi\frac{3}{2}$, or equivalently, $\frac{\pi}{ab-1} < \frac{2}{3}$. Thus, no matter the value of ϵ_1 or ϵ_2, the left (2.3) and right-hand (2.6) difference quotients have different signs and do not approach zero, which would fix the derivative at zero. Also, since $\lim_{m\to\infty}(ab)^m = \infty$, we see that the Weierstrass function f has no derivative at x_0. Interestingly, the difference in sign makes the derivatives approach infinity and negative-infinity, indicating that the derivatives will oscillate violently. Since x_0 was an arbitrary real number, it follows that f is nowhere differentiable on $\mathbb{R}$. □

3. Everywhere Continuity

Five terms with their definitions will be presented here. These terms are needed to prove that the Weierstrass function is everywhere continuous. The upper bound of a subset $A \subseteq \mathbb{R}$ is a number $\beta \in \mathbb{R}$ such that if for every $a \in A$, then $a \leq \beta$. The least upper bound, or the *supremum*, of a subset of real numbers A is a number α satisfying the following properties:

i: α is an upper bound of A.
ii: If $\gamma < \alpha$, then γ is not an upper bound of A.

Let $\{f_n\}$ be a sequence of functions, and let $n \in \mathbb{N}$. Then $\{f_n\}$ *converges uniformly* on an interval $A \subseteq \mathbb{R}$ to a function f if for every $\epsilon > 0$, there exists an integer N such that, if $n \geq N$, then $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in A$ [1]. Lastly, the *Cauchy criterion* states that a sequence of functions $\{f_n\}$ defined on $I \subseteq \mathbb{R}$ converges uniformly on I if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, and $x \in I$ implies that $|f_n(x) - f_m(x)| \leq \epsilon$ [1].

These two theorems and a corresponding corollary are used to prove the continuity of the Weierstrass function. The proofs of these statements are included.

Theorem 3 ([1] Weierstrass M-Test)**.** *Suppose $\{f_n\}$ is a sequence of functions defined on an interval $I \subseteq \mathbb{R}$, where $n \in \mathbb{N}$. If $|f_n(x)| \leq M_n$, then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.*

Proof. By assumption, $\sum M_n$ converges. Then, because $|f_n(x)| \leq M_n$, it follows that

$$\left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} M_i \leq \epsilon,$$

where $\epsilon > 0$ is arbitrary, $x \in I$, and m and n are large enough. It follows from the Cauchy criterion for uniform convergence that $\sum f_n(x)$ is uniformly convergent. □

Theorem 4 ([1])**.** *Suppose $\{f_n\}$ is a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$, where $n \in \mathbb{N}$. If f_n converges uniformly to some function f, then f is continuous on I.*

Proof. Let $x, y \in I$. Since $\{f_n\}$ converges uniformly to f, for all $\epsilon > 0$ there exists an $n \in \mathbb{Z}$ such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all $x \in I$. Since every f_n is continuous, it follows that there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Then,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon,$$

showing that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$, making f continuous. □

Corollary 1 ([3])**.** *Suppose $\{g_k\}$ is a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$, where $k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} g_k$ converges uniformly to a function f on I, then f is continuous on I.*

Proof. Let $f_n = \sum_{k=1}^{n} g_k$ be the n^{th} partial sum of functions from $\{g_k\}$. Note that each partial sum f_n is a continuous function because it is a finite sum of continuous functions. Saying $\sum g_k$ converges uniformly to f is equivalent to saying that

$$\lim_{n\to\infty} f_n = f.$$

Then, because $\{f_n\}$ converges uniformly to a function f, it follows that f is continuous by the Weierstrass M-Test. □

3.1. [3] **The Weierstrass Function is Everywhere Continuous.**

Proof. Consider the fact that, because $a \in (0, 1)$, the geometric series $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty$. Because $\sup_{x\in\mathbb{R}} |\cos(b^n \pi x)| \leq 1$, we have that $\sup_{x\in\mathbb{R}} |a^n \cos(b^n \pi x)| \leq a^n$. Thus, by the Weierstrass M-Test, the Weierstrass function is uniformly convergent for all real numbers. Also, by Corollary 1, the Weierstrass function is continuous everywhere on $\mathbb{R}$. □

4. Conclusions

The Weierstrass function presents an interesting counterexample to the converse of one of the most basic properties of differential calculus, i.e. that it is everywhere continuous but nowhere differentiable. We showed, through derivation, that the Weierstrass function cannot be differentiated at any point on the real line. Also, using a few theorems and definitions from higher analysis, we showed that it is everywhere continuous on the real line.

5. Acknowledgements

The creation of this report would not be possible without the creation of the Undergraduate Mathematics Seminar at Ohio University. Specifically, many thanks are due to Dr. Martin Mohlenkamp for his insightful comments and patience when reading multiple drafts of this report.

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