Discrete Dynamical Systems: The Linear, the Nonlinear, and the Chaotic Part I

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What is a dynamical system? In Words:

In order to define a dynamical system, we need to specify variables, the permissible values (states) they can take, and how these variables will change over time.

The primary focus of the subject is on the **qualitative features** of the **long-term behavior** of the system.

What is a dynamical system? In Symbols:

In order to define a dynamical system, we need to specify:

- A state space X,
- a time line $\mathbb{T}$ (usually $\mathbb{N} = \{0, 1, 2, \dots\}, \mathbb{Z}, [0, \infty)$, or $\mathbb{R}$),
- a time evolution function $\varphi : X \times \mathbb{T} \to X$ such that for all $x \in X$ and $t_1, t_2 \in \mathbb{T}$
  - $\varphi(x, 0) = x$
  - $\varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2)$.

The function $\varphi(x, \cdot)$ is called the **trajectory of (initial state)** x**,**

its restriction to positive times is the **forward trajectory of** x**,**

the set $O^+(x) := \{\varphi(x, t) : \ t \in \mathbb{T}, \ t \geq 0\}$ is called the **forward orbit of** x**,**

the set $O(x) := \{\varphi(x, t) : \ t \in \mathbb{T}\}$ is called the **(full) orbit of** x**.**

The focus of the subject is on studying orbits and the behavior of trajectories when $t \to \infty$.

Example 1: A population of rabbits triples every year

How can we model this real-world situation as a dynamical system?

Time: $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{Z}$ come to mind.

Let $x(t)$ denote the number of rabbits in year t.

The information given tells us that the time evolution function satisfies $\varphi(x, 1) = 3x$, and, more generally, $\varphi(x, t) = 3^t x$.

Assume, for example, that $x(0) = 9$.

Then the forward trajectory is the **sequence** $(9, 27, 81, 243, \dots)$.

The forward orbit is the **set** $\{9, 27, 81, 243, \dots\}$.

We can also extrapolate into the past:

Then the full trajectory is the **two-sided sequence** $(\dots 1, 3, 9, 27, 81, 243, \dots)$. Oops! How does one rabbit multiply?

The full orbit is the **set** $\{\dots 1/3, 1, 3, 9, 27, 81, 243, \dots\}$.

Oops! What is one third of a rabbit?

Example 1 continued: How should we choose X?

We are in a quandary: If we take $X = [0, \infty)$ for our state space, the system becomes **time-reversible,** which is a mathematically nice property. But only states in $\mathbb{N}$ make biological sense.

The best way out is to choose $X = \mathbb{R}$ for the state space and $\mathbb{T} = \mathbb{Z}$ for the time line. This will give our dynamical system the nicest possible properties.

We then call $\mathbb{N}$ the **biologically feasible region.**

This is again a judicious compromise between choosing a set with nice mathematical properties and biological realism. We conveniently ignore the fact that 1 rabbit cannot multiply all by itself and 10^{30} rabbits won't fit anywhere on earth.

The information given specifies a **difference equation**

$x_{t+1} = F(x_t)$ or $x(t+1) = F(x(t))$,

where $F : X \to X$ is the **map** $F(x) = 3x$, and x_t or $x(t)$ are often used for the state of the system at time t.

Discrete-Time Dynamical Systems

A (deterministic) **discrete-time dynamical system** is a pair (X, F) such that

- The state space X is a topological space.
- $F : X \to X$ is a continuous map.

The time evolution function is then given by

$\varphi(x, t) = F^t(x)$.

When F is a homeomorphism we will write (X, T) instead of (X, F). The system then becomes time-reversible and we take $\mathbb{T} = \mathbb{Z}$ as the time line.

When we write (X, F), we implicitly assume that $\mathbb{T} = \mathbb{N}$ is the time line.

(Forward) invariant sets

Consider a discrete dynamical system (X, T).

A subset $Y \subseteq X$ is **forward invariant** if

$T(y) \in Y$ for all $y \in Y$,

and is **invariant** if

both $T(y) \in Y$ and $T^{-1}(y) \in Y$ for all $y \in Y$.

An $x \in X$ is a **steady state** or **equilibrium** if $\{x\}$ is (forward) invariant.

In our Example 1 of the rabbits, the biologically feasible region $\mathbb{N}$ is forward invariant but not invariant,

the sets $(0, \infty), [0, \infty), (-\infty, 0)$ and $(-\infty, 0]$ are all invariant,

and $x = 0$ (extinction) is the only steady state.

The good: Linear systems

In our Example 1 of the rabbits, the state space X is a topological vector space and the map

$T(x) = 3x$ is linear.

We were able to immediately find a formula for the time evolution function and determine what is going on in the system.

In general, as we will see shortly, **linear systems** (X, F) or (X, T) are relatively easy to study in this sense. They are the "good" ones.

Example 2: A nonlinear system

Consider the system $((\mathbb{Z}^+)^2, F_e)$, where $(\mathbb{Z}^+)^2$ is the set of ordered pairs (x, y) of positive integers and

$$F_e(x, y) = (x, y - x) \quad \text{if} \quad x < y;$$
$$F_e(x, y) = (x - y, y) \quad \text{if} \quad x > y;$$
$$F_e(x, y) = (x, y) \quad \text{if} \quad x = y.$$

Can you write a formula for $\varphi((x, y), t)$? Good luck!

What happens in this system in the long run?

After finitely many steps, the system reaches the steady state $(gcd(x(0), y(0)), gcd(x(0), y(0)))$. This system embodies **Euclid's algorithm** for computing $gcd(x, y)$.

Each pair (x, x) is a steady state.

We can think of each set $\{(x, x)\}$ as an **attractor** that will be eventually **reached** from each initial state in its **basin of attraction.**

Example 3: Another simple nonlinear system

Consider the system $(\mathbb{Z}^+, F_c)$, where

$$F_c(n) = \frac{n}{2} \quad \text{if} \quad n \quad \text{is even;}$$
$$F_c(n) = 3n + 1 \quad \text{if} \quad n \quad \text{is odd.} \qquad (1)$$

Notice that this system is not time-reversible as

$F_c(8) = \frac{8}{2} = 4 = F_c(1) = 3 \cdot 1 + 1.$

What happens in this system in the long run?

Example 3 continued: Some observations

$$F_c(n) = \frac{n}{2} \quad \text{if} \quad n \quad \text{is even;}$$
$$F_c(n) = 3n + 1 \quad \text{if} \quad n \quad \text{is odd.} \tag{2}$$

- There are no steady states.
- $\{1, 2, 4\}$ is a **periodic orbit**.
- This periodic orbit is a attractor whose basin of attraction comprises at least all numbers up to 5.764×10^{18}.

The famous **Collatz Conjecture** states that $\{1, 2, 4\}$ is **globally attracting,** that is, its basin of attraction comprises all positive integers. For more info see
http://en.wikipedia.org/wiki/Collatz_conjecture.

These systems are of the form $(\mathbb{R}, F)$, where $F(x) = \lambda x$ and λ is a constant.

- Time-reversible iff $\lambda \neq 0$.
- 0 is always an equilibrium and is the only one iff $\lambda \neq 1$.
- If $\lambda = 1$, then every state is an equilibrium.
- If $\lambda = -1$, then every state except 0 belongs a periodic orbit of length 2; we also say that every nonzero state is a **periodic point of (minimal) period 2**.

Stability of the equilibrium at 0

Let $F(x) = \lambda x$.

What happens if the initial state is near, but not exactly at equilibrium?

- If $|\lambda| < 1$, then $\lim_{t\to\infty} x(t) = 0$ for all trajectories. This means that the equilibrium at 0 is **globally asymptotically stable** in this case.
- If $|\lambda| > 1$, then $\lim_{t\to\infty} |x(t) - 0| = \infty$ for all trajectories. This implies that the equilibrium at 0 is **unstable** in this case.
- If $|\lambda| = 1$, then for every $\varepsilon > 0$ there exists $\delta > 0$ ($\delta = \varepsilon$ works) such that $|x(t) - 0| < \varepsilon$ for all t as long as $|x(0) - 0| < \delta$. The equilibrium at 0 is **Lyapunov stable** in this case, but not asymptotically stable.

A few remarks on these notions

- **Asymptotic stability** means that trajectories will approach the equilibrium as $t \to \infty$. But in contrast to Examples 2 and 3, **usually** they will **not reach** the equilibrium in finite time.
- **Global** asymptotic stability means that **all** trajectories will approach the equilibrium; **local** asymptotic stability means that trajectories that start **sufficiently close** to the equilibrium will approach it, without ever getting too far away. There can be many locally stable equilibria, but if an equilibrium is globally stable, it must be the unique equilibrium.
- **Lyapunov stability** roughly means that trajectories that start near the equilibrium will stay close to it.
- An equilibrium is **unstable** iff it is **not** Lyapunov stable. This means that **at least some** trajectories that start arbitrarily close to the equilibrium will move away from it by a fixed positive minimal distance.
- Curiously enough, global asymptotic stability does **not always** imply local asymptotic stability.

Example 4: Fewer Rabbits

Consider the rabbit population of Example 1, but assume that each year a fox takes 100 rabbits.

How can we model this as a dynamical system?

We can again construct a discrete dynamical system $(\mathbb{R}, T)$.
What should the formula for T be?

Not enough information given!

Option 1: The fox feeds only after breeding takes place.

Then $T(x) = 3x - 100$. Here $x^* = 50$ is the unique equilibrium.

Option 2: The fox takes 40 rabbits before breeding season and 60 rabbits after breeding season.

Then $T(x) = 3(x - 40) - 60 = 3x - 180$.
Here $x^* = 90$ is the unique equilibrium.

Option 3: Ask a biologist for more data first!

In each case we get an **affine system.**

Multi-dimensional linear and affine systems

We will now consider systems $(\mathbb{R}^n, F)$. For simplicity we will mostly write x instead of $\vec{x}$.

- Such a system is **linear** if $F(x) = Mx$ for some $n \times n$ matrix M.
- The zero vector $\vec{0}$ is always an equilibrium, and it is unique iff $M - I$ is invertible, that is, iff 1 is not an eigenvalue of M.
- Such a system is **affine** if $F(x) = Mx + b$ for some vector b.
- If 1 is not an eigenvalue of M, then this system has a unique equilibrium $x^* = (I - M)^{-1}b$.
- If we introduce a new variable $y = x - x^*$, then the dynamics in terms of the new variable this system is the same as $(\mathbb{R}^n, My)$. Thus the study of affine systems reduces to the study of linear systems.
- What can we say about the stability of the equilibrium $\vec{0}$ in linear systems?

Case 1: M is diagonalizable

Assume that M has a full set of (n linearly independent) eigenvectors with real eigenvalues.

Then M is diagonalizable, which means that for a suitable choice of basis the function F can be written as

$F(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$,

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M. This gives

$F^t(x_1, \ldots, x_n) = (\lambda_1^t x_1, \ldots, \lambda_n^t x_n)$. Thus

- $\vec{0}$ is (locally and globally) asymptotically stable iff $\max |\lambda_i| < 1$,
- $\vec{0}$ is unstable if $\max |\lambda_i| > 1$,
- $\vec{0}$ is Lyapunov stable iff $\max |\lambda_i| \leq 1$.

The first two items are always true when all eigenvalues of M are real, the last item may fail if 1 or -1 is a repeated eigenvalue.

Case 2: M has a complex eigenvalue

Assume that M has conjugate complex eigenvalues $\lambda_1 = (r, \Theta), \lambda_2 = (r, -\Theta)$.

Then for a suitable choice of basis the function F can be written as

$$F(x_1, x_2, \ldots) = F(r_0 \cos\alpha, r_0 \sin\alpha, \ldots) = (rr_0 \cos(\alpha+\Theta), rr_0 \sin(\alpha+\Theta), \ldots).$$

This gives

$$F^t(x_1, x_2, \ldots) = (r^t r_0 \cos(\alpha + t\Theta), r^t r_0 \sin(\alpha + t\Theta), \ldots).$$

Now the (x_1, x_2)-plane is invariant, and **for initial conditions in this plane:**

- The trajectory spirals into $\vec{0}$ iff $r < 1$.
- The trajectory spirals out to infinity iff $r > 1$,
- The trajectory is confined to the circle with radius r_0 iff $r = 1$. It may be periodic with arbitrary period or aperiodic, depending on Θ.

The upshot

Consider any linear system $(\mathbb{R}^n, Mx)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of M.

Regardless of wether the eigenvalues of M are real or complex:

- $\vec{0}$ is (locally and globally) asymptotically stable iff $\max |\lambda_i| < 1$,
- $\vec{0}$ is unstable if $\max |\lambda_i| > 1$,
- If none of the eigenvalues is repeated, then $\vec{0}$ is Lyapunov stable iff $\max |\lambda_i| \leq 1$.

Another way of looking at this result

Consider any linear system $(\mathbb{R}^n, Mx)$ and **assume that M has a full set of (real or complex) eigenvectors.**

- Let E_s be the linear span of all eigenvectors whose eigenvalues satisfy $|\lambda_i| < 1$.
  This is called the **stable subspace.**
  All trajectories that start in E_s approach $\vec{0}$ as $t \to \infty$.
- Let E_u be the linear span of all eigenvectors whose eigenvalues satisfy $|\lambda_i| > 1$.
  This is called the **unstable subspace.**
  All trajectories that start in E_u will approach $\vec{0}$ as $t \to -\infty$.
- Let E_c be the linear span of all eigenvectors whose eigenvalue satisfies $|\lambda_i| = 1$.
  This is called the **center subspace.**
  All trajectories that start in E_c stay at a fixed distance of $\vec{0}$.

Each of the subspaces E_s, E_u, E_c is an invariant set.

An application: Linear stability analysis

But the most interesting systems are not linear!

True enough. But consider an equilibrium x^* in system (X, T) and assume that near x^* we can parametrize the system by vectors in $\mathbb{R}^n$ so that T is a diffeomorphism.

For simplicity of notation we may assume here that $X = \mathbb{R}^n$. Moreover, using essentially the same trick as for reducing affine to linear systems, we may wlog assume that $x^* = \vec{0}$.

Since T is differentiable, there exists a matrix M such that

$T(x) = Mx + R(x)$, where $\lim_{\|x\| \to 0} \frac{\|R(x)\|}{\|x\|} = 0$.

The system $(\mathbb{R}^n, Mx)$ is called **the linearization of (X, F) at x^*.**

- If all eigenvalues of M satisfy $|\lambda_i| < 1$,
  then x^* is **locally** asymptotically stable in (X, T).
- If there exists at least one eigenvalue λ_i with $|\lambda_i| > 1$,
  then x^* is unstable in (X, T).

Why does this work?

Assume that x^* is a **hyperbolic equilibrium,** which means that for the system $(\mathbb{R}^n, Mx)$ of the previous slide we have $E_c = \{\vec{0}\}$.

Then there exists a continuous map $h : \mathbb{R}^n \to X$, called a **local conjugacy,** that slightly deforms the geometry near $\vec{0}$ so that it changes trajectories of $(\mathbb{R}^n, Mx)$ near $\vec{0}$ into trajectories of (X, T) near x^*.

In particular, h will bend E_s into the **stable manifold** (of the same dimension as E_s) at x^* in (X, T).
All trajectories that start in the stable manifold will approach x^* as $t \to \infty$.

Similarly, h will bend E_u into the **unstable manifold** (of the same dimension as E_u) at x^* in (X, T).
All trajectories that start in the stable manifold will approach x^* as $t \to -\infty$, and, in particular, will (initially) move away from x^*.

Thus if E_u has dimension at least 1, x^* must be unstable in (X, T).

Otherwise E_s must have dimension n (by hyperbolicity), and the stable manifold will contain a **neighborhood of** x^***,** so that x^* is **locally** asymptotically stable in (X, T).

But what if?

- But what if the equilibrium x^* is not hyperbolic?
  - If M has at least one eigenvalue with $|\lambda_i| > 1$, then we can still conclude that x^* is unstable in (X, T).
  - If $\max |\lambda_i| = 1$, then all bets are off. There is no general theorem for this case.
- Where does this funny name "hyperbolic equilibrium" come from?
  - Some mathematicians just love hyperbole. But seriously ...
- And even if the equilibrium is hyperbolic but has both a stable and unstable manifold of positive dimensions, what happens to a trajectory that start outside the unions of these sets?
  - **See you next Tuesday!**