Marvin and Marilyn Go on a Diet

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A health-conscious couple

Marilyn and Marvin decide to go on Dr. Losit's new scientifically proven diet. According to this expert, they should eat exactly 50 grams of sugar, exactly 300 grams of protein, and exactly 100 grams of fat per day, and restrict themselves to one meal per day prepared from a mixture of a wide range of products offered by his company.

Being on a budget and having somewhat different tastes, they decide to purchase Losit-Quick (strawberry taste, which they both like), Losit-Fast (Marylin's beloved watermelon taste that Marvin cannot stand), and Losit-Easy (beer-flavored for Marvin, detested by Marylin).

The products are in, now what?

After receiving the products in the mail and reading the fine print on the labels, they discover that:

- One serving of Losit-Quick contains 20 grams of sugar, 200 grams of protein, and 20 grams of fat.
- One serving of Losit-Fast contains 15 grams of sugar, 50 grams of protein, and 40 grams of fat.
- One serving of Losit-Easy contains 25 grams of sugar, 150 grams of protein, and 60 grams of fat.

How should they compose their meals if they want to strictly follow the guidelines?

Each of our protagonists sits down with a pencil and paper and tries to figure it out.

Marilyn's reasoning

Marilyn reasons as follows: I will need x_1 servings of Losit-Quick and x_2 servings of Losit-Fast.

The numbers x_1, x_2 need to be chosen in such a way that the total amounts of sugar, protein, and fat add up to exactly the recommended values.

They must be solutions of the following system of linear equations, where all quantities are expressed in grams:

$$\begin{aligned} 20x_1 + 15x_2 &= 50 \quad \text{(the total amount of sugar)} \\ 200x_1 + 50x_2 &= 300 \quad \text{(the total amount of protein)} \\ 20x_1 + 40x_2 &= 100 \quad \text{(the total amount of fat)} \end{aligned}$$

She is really good at Gaussian elimination and uses it to quickly solve the system.

Marilyn's solution

$$\begin{bmatrix} 20 & 15 & 50 \\ 200 & 50 & 300 \\ 20 & 40 & 100 \end{bmatrix} \xrightarrow{\text{subtract } 10(\text{row } 1) \text{ from row } 2} \begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 20 & 40 & 100 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 20 & 40 & 100 \end{bmatrix} \xrightarrow{\text{subtract row } 1 \text{ from row } 3} \begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 15 & 50 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{\text{divide row } 1 \text{ by } 20} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix}$$

Marilyn's solution, continued

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & -100 & -200 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{\text{divide row 2 by -100}} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 25 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 25 & 50 \end{bmatrix} \xrightarrow{\text{divide row 3 by 25}} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{subtract row 2 from row 3}} \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Marilyn's starts cooking

Marilyn has found the augmented matrix

$$\mathbf{A}^{\vec{b}} = \begin{bmatrix} 1 & 0.75 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

of the equivalent system

$$\begin{aligned} x_1 + 0.75x_2 &= 2.5 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

This tells her that she needs to mix $x_2 = 2$ servings of Losit-Fast with $x_1 = 1$ serving of Losit-Quick.
She starts preparing her meal.

In the meantime Marvin ...

who isn't a big fan of Gaussian elimination tried randomly guessing an appropriate mixture of Losit-Fast and Losit-Easy, without much success.

He likes vector notation though, and reasons as follows:
Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of amounts of sugar, protein, and fat in one serving of Losit-Quick, Losit-Easy, and in the meal that I need to compose. Then

$$\vec{\mathbf{v}}_Q = \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} \qquad \vec{\mathbf{v}}_E = \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} \qquad \vec{\mathbf{v}}_M = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix}$$

I need to find scalars d_Q, d_E such that $\vec{\mathbf{v}}_M$ is the **linear combination**

$d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M,$

where d_Q, d_E are the numbers of servings of Losit-Quick and Losit-Easy that I need for my meal.

Marvin is getting hungry

It remains to find the actual numbers d_Q, d_E such that $\vec{\mathbf{v}}_M$ is the linear combination

$d_Q\vec{\mathbf{v}}_Q + d_E\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$. Hmmm. ????

Eventually hunger prevails over pride and Marvin decides that his notation and the phrase **linear combination** look sufficiently impressive so that it wouldn't be too embarassing to ask:

Hey, sweetheart, I have practically figured it out. Could you just give me a hand with these boring calculations?

Marilyn tells him: Sorry, bud. You will need to add some of my Losit-Fast.

Homework 22: Is Marilyn right or just being nasty?

Homework 23: If Marilyn is right, what is the minimum amount of Losit-Fast that Marvin will need to digest? What is the maximum amount of Losit-Easy that he can add?

Solution for Homework 22

Marilyn argues as follows: You will need x_1 servings of Losit-Quick and x_2 servings of Losit-Easy.

The numbers x_1, x_2 need to be chosen in such a way that the total amounts of sugar, protein, and fat add up to exactly the recommended values.

They must be solutions of the following system of linear equations, where all quantities are expressed in grams:

$$
\begin{aligned}
20x_1 + 25x_2 &= 50 \quad \text{(the total amount of sugar)}\\
200x_1 + 150x_2 &= 300 \quad \text{(the total amount of protein)}\\
20x_1 + 60x_2 &= 100 \quad \text{(the total amount of fat)}
\end{aligned}
$$

She now does a Gaussian elimination for this system:

Gaussian elimination for Homework 22

$$\begin{bmatrix} 20 & 25 & 50 \\ 200 & 150 & 300 \\ 20 & 60 & 100 \end{bmatrix} \underset{\longrightarrow}{\text{subtract 10(row 1) from row 2}} \begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 20 & 60 & 100 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 20 & 60 & 100 \end{bmatrix} \underset{\longrightarrow}{\text{subtract row 1 from row 3}} \begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 25 & 50 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix} \underset{\longrightarrow}{\text{divide row 1 by 20}} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix}$$

Gaussian elimination for Homework 22, continued

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & -100 & -200 \\ 0 & 35 & 50 \end{bmatrix} \xrightarrow{\text{divide row 2 by -100}} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 35 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 35 & 50 \end{bmatrix} \xrightarrow{\text{subtract 35(row 2) from row 3}} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & -20 \end{bmatrix} \xrightarrow{\text{divide row 3 by -20}} \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Marvin is out of luck

Marilyn has found the augmented matrix

$$\mathbf{A}^{\vec{\mathbf{b}}} = \begin{bmatrix} 1 & 1.25 & 2.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

of the equivalent system

$$\begin{aligned} x_1 + 1.25x_2 &= 2.5 \\ x_2 &= 2 \\ 0 &= 1 \end{aligned}$$

This system is **inconsistent.** Marvin, you cannot compose his meal without adding some Losit-Fast.

Or so Marilyn says.

Marvin is not conceding yet

Marvin: (looking dejected) Ok, we can't figure it out with your Gaussian elimination. But what if we approach the problem from my angle and look for coefficients of a linear combination instead?

Marilyn: Er, well, ... I don't really understand your linear combination stuff. Can you explain it to me?

Marvin: (becomes animated again) This is really quite simple. It works like this: If you have any vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ and another vector $\vec{\mathbf{w}}$, then $\vec{\mathbf{w}}$ is a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if, and only if, there exist numbers (or as the textbook says, scalars) $d_1, d_2, \dots, d_n$ such that

$$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \dots + d_n\vec{\mathbf{v}}_n = \vec{\mathbf{w}}.$$

Marilyn: Can you give me an example?

Marvin: Sure. Let $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E, \vec{\mathbf{v}}_M$ be the vectors of amounts of sugar, protein, and fat in one serving of Losit-Quick, Losit-Fast, Losit-Easy, and in the meal that each of us needs to compose.

Marvin: Then $\vec{\mathbf{v}}_Q + 2\vec{\mathbf{v}}_F = \vec{\mathbf{v}}_M$. So if we let $d_1 = 1, d_2 = 2$, then we see that $\vec{\mathbf{v}}_M$ is a linear combination of $\vec{\mathbf{v}}_Q$ and $\vec{\mathbf{v}}_F$.

Marvin continues: We can look at this coordinate by coordinate.

$$\vec{\mathbf{v}}_Q = \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} \qquad \vec{\mathbf{v}}_F = \begin{bmatrix} 15 \\ 50 \\ 40 \end{bmatrix} \qquad \vec{\mathbf{v}}_E = \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} \qquad \vec{\mathbf{v}}_M = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix}$$

So we have

$$d_1\vec{\mathbf{v}}_Q + d_2\vec{\mathbf{v}}_F = d_1 \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} + d_2 \begin{bmatrix} 15 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} d_1 20 + d_2 15 \\ d_1 200 + d_2 50 \\ d_1 20 + d_2 40 \end{bmatrix} = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix} = \vec{\mathbf{v}}_M$$

When $d_1 = 1$ and $d_2 = 2$ it all adds up.

Marilyn: Wow! Can you write this for me as a system of linear equations? You know, I find systems easier to understand.

Marvin translates it into systems

Marvin: When we look at the coordinates that give us the amounts of sugar, protein, and fat one by one, we get the following system of linear equations:

$$
\begin{aligned}
20d_1 + 15d_2 &= 50 \quad \text{(the total amount of sugar)}\\
200d_1 + 50d_2 &= 300 \quad \text{(the total amount of protein)}\\
20d_1 + 40d_2 &= 100 \quad \text{(the total amount of fat)}
\end{aligned}
$$

Marilyn: Could I use x_1, x_2 instead of d_1, d_2 here?

Marvin: Absolutely! You see, it doesn't matter which letters you use for your variables. These are just names.

Marilyn: So the coefficients $d_1 = 1, d_2 = 2$ that you showed me here are precisely the numbers that I got from my solution of the system by Gaussian elimination?

Marvin: Right!

Marvin's meal

Marilyn: (with a sly smile) So how about your meal?

Marvin: I need coefficients d_1, d_2 such that $d_1\vec{\mathbf{v}}_Q + d_2\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M$. We can look at this coordinate by coordinate.

$$d_1\vec{\mathbf{v}}_Q + d_2\vec{\mathbf{v}}_E = d_1 \begin{bmatrix} 20 \\ 200 \\ 20 \end{bmatrix} + d_2 \begin{bmatrix} 25 \\ 150 \\ 60 \end{bmatrix} = \begin{bmatrix} d_1 20 + d_2 25 \\ d_1 200 + d_2 150 \\ d_1 20 + d_2 60 \end{bmatrix} = \begin{bmatrix} 50 \\ 300 \\ 100 \end{bmatrix} = \vec{\mathbf{v}}_M$$

Marvin: Since you like systems better ...

Marilyn: And I like x_1, x_2 better than d_1, d_2.

Marvin: (ever the gentleman) We will write it as

$$\begin{aligned} 20x_1 + 25x_2 &= 50 \quad \text{(the total amount of sugar)} \\ 200x_1 + 150x_2 &= 300 \quad \text{(the total amount of protein)} \\ 20x_1 + 60x_2 &= 100 \quad \text{(the total amount of fat)} \end{aligned}$$

(A moment of profound silence ...)

Marvin: (Deep breath. Takes it like a man.) Now you can see that my approach is really the same as yours, only using a more condensed notation and different terminology. Since the system is inconsistent, we will need to use ingredient Losit-Fast.
We can find the serving sizes x_1, x_2, x_3 by solving the following system:

$$
\begin{aligned}
20x_1 + 15x_2 + 25x_3 &= 50 \quad \text{(the total amount of sugar)} \\
200x_1 + 50x_2 + 150x_3 &= 300 \quad \text{(the total amount of protein)} \\
20x_1 + 40x_2 + 60x_3 &= 100 \quad \text{(the total amount of fat)}
\end{aligned}
$$

Marilyn interjects: By using Gaussian elimination.

Marvin: We can leave this an an exercise for the students. I can see right off the bat that the solution is $x_1 = 1, x_2 = 2, x_3 = 0$.

(It dawns upon Marvin what $x_3 = 0$ means.

No beer-flavored Losit-Easy.

Long and profound silence ...)

Marvin's meal, Plan C. Maybe.

Marilyn: (soothingly) Maybe there is a different solution?

Homework 24: Solve the system on the previous slide by using Gaussian elimination. Determine all solutions.

Marilyn: (gently) Should we try your approach?

Marvin: (with a glimmer of hope returning to his face) Good idea. Let's assume that $[y_1, y_2, y_3]^T$ is a **different solution.**

Then $\vec{\mathbf{v}}_M$ can be expressed as a linear combination of $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E$ in at least two different ways:

$$x_1\vec{\mathbf{v}}_Q + x_2\vec{\mathbf{v}}_F + x_3\vec{\mathbf{v}}_E = \vec{\mathbf{v}}_M = y_1\vec{\mathbf{v}}_Q + y_2\vec{\mathbf{v}}_F + y_3\vec{\mathbf{v}}_E.$$

Marilyn: When we subtract the right-hand side from the left-hand side, $\vec{\mathbf{v}}_M$ cancels and we get

$$(x_1 - y_1)\vec{\mathbf{v}}_Q + (x_2 - y_2)\vec{\mathbf{v}}_F + (x_3 - y_3)\vec{\mathbf{v}}_E = \vec{\mathbf{0}},$$

where $\vec{\mathbf{0}}$ is the zero vector that has entry 0 in all coordinates.

Marvin and Marilyn: What does this tells us about the meal?

Marvin's meal, Plan D.

Marvin: We can simplify the notation by writing the equation

$(x_1 - y_1)\vec{\mathbf{v}}_Q + (x_2 - y_2)\vec{\mathbf{v}}_F + (x_3 - y_3)\vec{\mathbf{v}}_E = \vec{\mathbf{0}}$, in the form

$c_1\vec{\mathbf{v}}_Q + c_2\vec{\mathbf{v}}_F + c_3\vec{\mathbf{v}}_E = \vec{\mathbf{0}}$,

where $c_1 = x_1 - y_1, c_2 = x_2 - y_2, c_3 = x_3 - y_3$.

Marilyn: If $[x_1, x_2, x_3]^T$ and $[y_1, y_2, y_3]^T$ are really different solutions, then not all of the constants c_1, c_2, c_3 can be zero.

Marvin: So this means a second solution can exist only if the vectors $\vec{\mathbf{v}}_Q, \vec{\mathbf{v}}_F, \vec{\mathbf{v}}_E$ are what is called **linearly dependent.**
But how can we find out whether or not this is the case?

Marilyn: We could use Gauss ... (senses a looming relationship crisis and bites her tongue) You know what, Prof. Just is going to teach us some great methods for determining linear dependence in a few days. So how about postponing the diet and having some burgers with fries

Marvin: and a couple of beers!