Emergence of global firing patterns in neuronal networks with random connectivities

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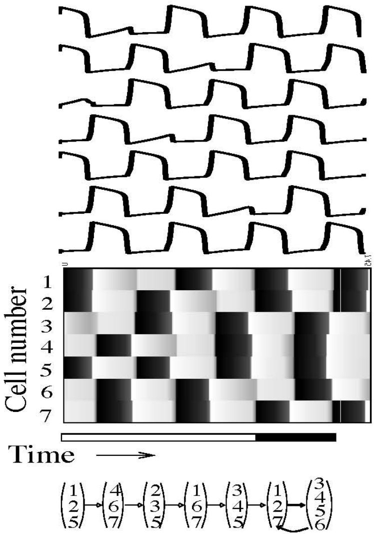
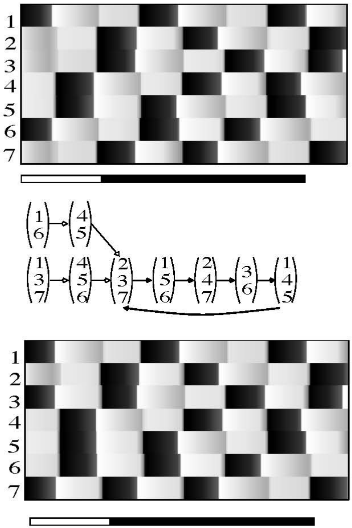
A problem from mathematical neuroscience

Some recordings from certain neuronal tissues (of real organisms) reveal the following pattern: Time seems to be partitioned into episodes with surprisingly sharp boundaries. During one episode, a group of neurons fires, while other neurons are at rest. In the next episode, a different group of neurons fires. Group membership may vary from episode to episode, a phenomenon called dynamic clustering.

How dynamic clustering looks like





What could drive dynamic clustering?

The graphs on the previous slide are based on simulations of an ODE model that was analyzed in

Terman D, Ahn S, Wang X, Just W, Physica D, 2008.

The model assumes a specific architecture that involves layers of excitatory and inhibitory neurons. Firing input between excitatory neurons is mediated by the inhibitory layer via a mechanism called post-inhibitory rebound.

- There is no central pacemaker in this model. We proved that the partitioning into eposodes on a global scale emerges spontaneously for a generic set of initial conditions.
- We also proved that the emerging dynamics of the excitatory neurons can be described by a discrete model that predicts, for all sufficiently large times which neurons will fire during the next episode.

Mathematical neuroscience for the rest of us

The following is true in at least some neuronal networks.

- Neurons fire or are at rest.
- After a neuron has fired, it has to go through a certain refractory period when it cannot fire.
- Neurons are connected via synapses. Through a given synapse, the presynaptic neuron may send firing input to the postsynaptic neuron.
- A neuron will fire when it has reached the end of its refractory period and when it receives firing input from a specified minimal number of other neurons.

This is of course way too simple ...

but let us build a class of simple models N of neuronal networks based on these facts.

Discrete dynamical system models $N(D)$

Let $D = ([n], A_D)$ be a digraph on $[n] = \{1, \dots, n\}$.

We describe here only the simplest case when all refractory periods and firing thresholds are 1.

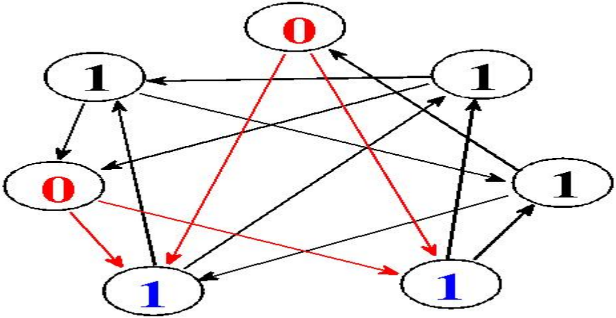
A state $\vec{s}(t)$ at the discrete time t is a vector:
$\vec{s}(t) = (s_1(t), \dots, s_n(t))$ where $s_i(t) \in \{0, 1\}$ for each i.
The state $s_i(t) = 0$ means neuron i fires at time t.

Dynamics of $N(D)$:

- If $s_i(t) < 1$, then $s_i(t+1) = s_i(t) + 1 = 1$.
- If $s_i(t) = 1$, and there exists at least one neuron j with $s_j(k) = 0$ and $< j, i > \ \in A_D$, then $s_i(t+1) = 0$.
- If $s_i(t) = 1$ and there does not exist a neuron j with $s_j(t) = 0$ and $< j, i > \ \in A_D$, then $s_i(t+1) = 1$.

$N(D)$ is a Boolean dynamical system.

An example



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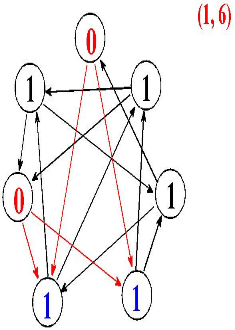
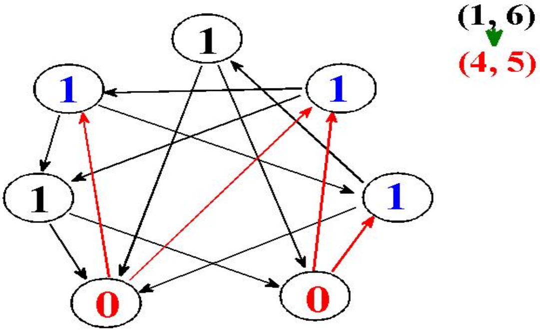
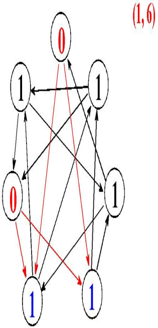
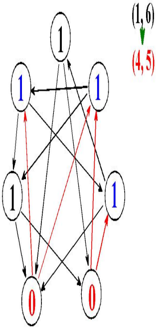
An example

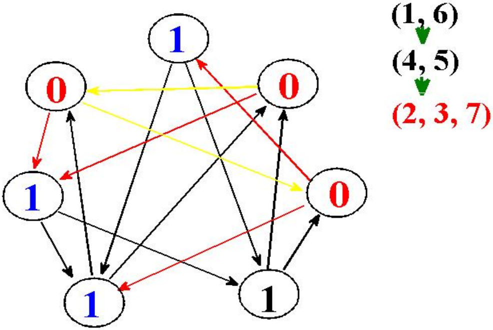




An example







Why study networks with random connectivities?

Amazing fact: There exists a little roundworm, *Caenorhabditis elegans,* with 302 neurons, for which each single synapse has been mapped!

But for other higher organisms our knowledge of the actual neuronal wiring is only very fragmentary. We may, however, have some information about global network parameters such as the degree distribution. For example, there are about 10^{12} neurons and 10^{15} synaptic connections in the human brain, which gives a mean degree of about 1000 for the network.

The architecture of actual neuronal networks has been shaped by evolution and to some extent by learning, both of which are **stochastic processes.** Thus it is reasonable to assume that the actual architecture exhibits features that are reasonably typical for a relevant probability distribution on digraphs.

The basic setup

Just W and Ahn S arXiv:1404.5536 (2014)

- Let π be a function that assigns to each positive integer n a probability $\pi(n)$.
- Randomly draw an Erdős-Rényi digraph D on $[n]$ where each potential arc is included with probability $\pi(n)$.
- Randomly draw an initial condition $\vec{s}(0)$ in the chosen network.
- Let α be the length of the attractor and let τ be the length of the transient of the trajectory of $\vec{s}(0)$.
- Explore how α and τ scale on average w.r.t. the number n of neurons.

Why are these scaling laws relevant?

Biological relevance: Dynamic clustering has been observed in some olfactory systems. Lenghts of attractors and transients are relevant to an ongoing debate about how odors are coded.

Mathematical relevance: Boolean systems can be roughly categorized as those with ordered dynamics and those with chaotic dynamics. The former are characterized (among other hallmarks) by relatively short transients and attractors; the latter by relatively long ones. The difference between "short" and "long" often corresponds to polynomial *vs.* exponential scaling with system size n.

The capability of the system to perform complex computations appears to require that its dynamics falls into the critical regime, right at the boundary between order and chaos.

α and τ for very sparse connectivities

Theorem (The subcritical case)

Assume $\pi(n) = \frac{c}{n}$ *with* $c < 1$. *Then*

(i) The lentgth α *of the attractor scales like* $O(1)$.

(ii) The length τ *of the transient scales like* $\Theta(\log n)$.

Thus the subcritical case exhibits hallmarks of highly ordered dynamics.

Theorem (The supercritical case)

There exists a constant c_{crit} with $1 \leq c_{crit} \leq 2$ such that if $\pi(n) = \frac{c}{n}$ for some fixed $c > c_{crit}$:

(ii) α scales like $O(1)$.

(iii) τ scales like $\Omega(\log n)$.

(iv) τ scales like $O(n^k)$ for some constant $k = k(c) > 0$.

Again, we observe hallmarks of highly ordered dynamics.

Why?

Conjecture 1: $c_{crit} = 1$. Simulations suggest as much.

Problem 2: Determine the precise scaling law for τ in the supercritical case.

Why? Emergence of the giant component

It is also (well) known that when $\pi(n)$ is increased from $\frac{c_\ell}{n}$ to $\frac{c^u}{n}$ for some $c_\ell < 1 < c^u$, then a so-called giant strongly connected component that comprises a fixed fraction of all nodes appears a.a.s. in the corresponding Erdős-Rényi digraph D.

There have been detailed studies of the expected structure of D in the so-called critical window where $\pi(n) \sim \frac{1}{n}$.

Eventually minimally cycling nodes

Definition

A node i is eventually minimally cycling if there are only finitely many times t with $s_i(t) = s_i(t+1) = 1$.

Intuitively, a node is eventually minimally cycling if from some time on it will always fire as soon as it has reached the end of its refractory period.

If the giant strongly connected component contains an eventually minimally cycling node (local property), then all of its nodes become eventually minimally cycling (global property).

The supercritical case revisited

Theorem (The supercritical case)

There exists a constant c_{crit} with $1 \leq c_{crit} \leq 2$ such that if $\pi(n) = \frac{c}{n}$ for some fixed $c > c_{crit}$:

(i) Asymptotically almost surely, all nodes in the giant component will be eventually minimally cycling.

(ii) α scales like $O(1)$.

(iii) τ scales like $\Omega(\log n)$.

(iv) τ scales like $O(n^k)$ for some constant $k = k(c) > 0$.

Theorem (Lower end of the critical case)

Assume $\pi(n) = \frac{1-n^{-\beta}}{n}$, *where* $0 < \beta < 1/4$. *Then with probability arbitrarily close to 1 as* $n \to \infty$

(i) τ *scales like* $O((\log n)n^{\beta})$.

(ii) τ *scales like* $\Omega(n^{\beta})$.

(iii) $\alpha \leq e^{\sqrt{n \ln n} + o(1)}$ *and thus scales subexponentially.*

(iv) $\alpha \geq e^{\Omega(\log n \log \log n)}$ *and hence scales faster than any polynomial function.*

We observe one hallmark of the critical regime for the dynamics.

What happens in the middle of the critical window?

Conjecture 2: For $\pi(n) = \frac{1}{n}$ both α and τ scale even faster.

Simulations studies indicate as much.

A rigorous derivation of scaling laws appears to require new tools.

General observations. Draw our own conclusions.

- We have seen theorems on global dynamics of deterministic finite dynamical systems with random connectivities.
- Many natural open problems remain.
- These theorems draw on the theory of random graphs. Some proofs merely quote well-known results, other proofs rely on new "customized" results, some open problems are likely to require development of new tools for the study of random structures.
- Each of these results or conjectures has a counterpart that can be phrased in terms of the global dynamics of the corresponding ODE systems that were mentioned at the beginning of this talk.
- Currently there is no established community of researchers who focus on the type of problems presented here, or a recognizable core body of mathematical tools for solving them.

Problem 0: What is c_{crit}, really?

We showed that $1 \leq c_{crit} \leq 2$.

We conjecture that $c_{crit} = 1$.

Simulation results indicate as much.

Open problems: An even bigger one

Define locally modified Erdős-Rényi (di)-graphs as follows:

- Consider an algorithm A that takes as input a (di)graph D on $[n]$ with some labeling of the vertices with a fixed set of labels, and outputs another labeled (di)graph $A(D)$ on $[n]$.
- The algorithm decides whether or not $< i, j >$ is an arc (edge) of $A(D)$ only based on the structure and labels of subgraph induced by all nodes that can be reached from i or j via a (directed) path of length $\leq N$, where N is fixed and does not depend on n.
- Let D be an Erdős-Rényi (di)graph.
- Generate the labels independently, with specified probabilities of assigning a given label.

This defines a family of distributions $A(D)$.

Problem 1: What global properties of Erdős-Rényi random (di)graphs carry over to such distributions?

More open problems

Problem 2: Find the exact scaling law for the length τ of the transient in the supercritical case, or at least narrow the gap between $\Omega(\log n)$ and $O(n^{k(c)})$.

Problem 3: Assume $\pi(n) = \frac{1-n^{-\beta}}{n}$, where $0 < \beta < 1/4$. Find the exact scaling law for the length τ of the transient.

At this time we know that it is between $\Omega(n^{\beta})$ and $O((\log n)n^{\beta})$.

Problem 4: Does there exist $\pi(n)$ such that $\tau(n)$ scales faster than any polynomial?

At this time we don't even know whether there exists $\pi(n)$ where $\tau(n)$ scales like $\Omega(n)$.

Problem 5: Does there exist, for any n, a network $N(D)$ on $[n]$ that contains any attractor of length $\alpha > g(n)$, where $g(n) \sim e^{\sqrt{n \ln n}+o(1)}$ is Landau's function?

Open problems for related systems

Problem 6: Investigate α and τ for analogous systems with larger firing thresholds.

We have some results, but a full characterization will require new methods.

Problem 7: Investigate the behavior of α and τ for other types of random connectivities.

Some empirical results indicate that the degree distributions in actual neuronal networks may be closer to scale-free than to normal. Thus making D a random scale-free network may be more relevant to neuroscience. But we had to start our investigations somewhere.

Problem 8: Try to generalize our results to systems with other types of rules for the firing of neurons.

Published work on connectivity D *vs.* dynamics of $N(D)$

- W. Just, S. Ahn, and D. Terman (2008); Minimal attractors in digraph system models of neuronal networks. *Physica D* **237**, 3186–3196.

  Two phase transitions for dense random connectivities.
- S. Ahn, Ph. D. Thesis (OSU) and S. Ahn and W. Just (2012); Digraphs vs. Dynamics in Discrete Models of Neuronal Networks. *Discrete and Continuous Dynamical Systems - Series B (DCDS-B)* **17**(5) 1365–1381.

  Characterizes possible dynamics for some basic connectivities.
- W. Just, S. Ahn, and D. Terman (2013); Neuronal Networks: A Discrete Model. In *Mathematical Concepts and Methods in Modern Biology.* R. Robeva and T. Hodge, eds., Academic Press, 2013, 179–211.

  Elementary introduction and overview. Suitable as basis for REU.
- W. Just and S. Ahn (2014); Lengths of attractors and transients in neuronal networks with random connectivities. *Preprint.* arXiv:1404.5536 A shortened journal version has been submitted.