NEW CHARACTERIZATION OF Σ-INJECTIVE MODULES

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Abstract. We provide a new characterization for an injective module to be Σ-injective.

1. INTRODUCTION

In his paper [4], Carl Faith introduced the concept of Σ-*injectivity* and defined an injective module M to be Σ-injective if every direct sum of copies of M is injective. It turns out that such an R-module M provides a good deal of information about the structure of a ring R. For example, R is right noetherian if and only if every injective right R-module is Σ-injective [5]. If R is an integral domain then the injective hull $E(R_R)$ of R is Σ-injective if and only if R is a right Ore domain [4]. Goursaud-Valette showed that if a ring R admits a faithful Σ-injective module then R is a right Goldie ring [6].

The following characterizations are well-known for an injective module to be Σ-injective.

Theorem 1. *(Cailleau [3], Faith [4]) For an injective module M_R, the following are equivalent:*

(1) M is Σ-injective.

(2) M is countably Σ-injective.

(3) R satisfies ACC on the the set of right ideals I of R that are annihilators of subsets of M.

(4) M is a direct sum of indecomposable Σ-injective modules.

The purpose of this paper is to provide the following new characterization for an injective module to be Σ-injective.

Theorem 2. *Let M_R be an injective module. Then the following statements are equivalent:*

(a) M is Σ-injective.

(b) There exists an infinite cardinal α such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules.

2000 *Mathematics Subject Classification.* 16D50, 16P40.

Key words and phrases. Injective modules, Σ-injective modules, essential extensions, right noetherian rings.

First author is deceased.

2. PRELIMINARIES

All rings considered in this paper have unity and all modules are right unital. We denote by $E(M)$, the injective hull of M. We shall write $N \subseteq_e M$ whenever N is an essential submodule of M. A submodule L of M is called an essential closure of a submodule N of M if it is a maximal essential extension of N in M. A submodule K of M is called a complement if there exists a submodule U of M such that K is maximal with respect to the property that $K \cap U = 0$. Given a cardinal α and a module N, we denote by $N^{(\alpha)}$ the direct sum of α copies of the module N. A module N is said to be Σ-injective provided that $N^{(\alpha)}$ is injective for any cardinal α. We say that the Goldie dimension of N with respect to U, $G\dim_U(N)$, is less than or equal to n, if for any independent family $\{V_j : j \in \mathcal{J}\}$ of nonzero submodules of N such that each V_j is isomorphic to a submodule of U, we have that $|\mathcal{J}| \leq n$. Next, the notation $G\dim_U(N) < \infty$ means that $G\dim_U(N) \leq n$ for some positive integer n. A module N is said to be $q.f.d.$ relative to U if for any factor module $\bar{N}$ of N, $G\dim_U(\bar{N}) < \infty$. We say R is right $q.f.d.$ relative to M if R_R is $q.f.d.$ relative to M.

We first start with a key lemma.

Lemma 3. *Let M be an injective module and suppose there exists an infinite cardinal α such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules. Then*

(a) Given a direct sum $G = \oplus_{i\in\mathbb{N}} M_i$, $M_i \cong M$, and nonzero injective submodules V_i of M_i, $i \in \mathbb{N}$, there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ and nonzero injective submodules $V_j^{'} \subseteq V_j$, $j \in \mathcal{J}$, such that $\oplus_{j\in\mathcal{J}} V_j^{'}$ is injective.

In particular, if $\{V_i : i \in \mathbb{N}\}$ is an independent family of uniform injective submodules of M then $\oplus_{j\in\mathcal{J}} V_j$ is injective for some infinite subset $\mathcal{J} \subseteq \mathbb{N}$.

(b) R is right $q.f.d.$ relative to M.

Proof. (a) Set $E = E(G)$. Since V_i is an injective submodule of M_i, $M_i = V_i \oplus M_i^{'}$ for some submodule $M_i^{'} \subseteq M_i$. Therefore, $G = (\oplus_{i\in\mathbb{N}} V_i) \oplus (\oplus_{i\in\mathbb{N}} M_i^{'})$. Let H and $H^{'}$ be essential closures of $\oplus_{i\in\mathbb{N}} V_i$ and $\oplus_{i\in\mathbb{N}} M_i^{'}$ in E, respectively. Clearly, $E = H \oplus H^{'}$. If $\oplus_{i\in\mathbb{N}} V_i = H$, then there is nothing to prove.

Consider now the case when $\oplus_{i\in\mathbb{N}} V_i \neq H$. Pick $x \in H \setminus \oplus_{i\in\mathbb{N}} V_i$. Let Q be a submodule of H maximal with respect to the properties that $\oplus_{i\in\mathbb{N}} V_i \subseteq Q$ and $x \notin Q$. Set $P = Q \oplus H^{'}$ and note that $E/P = (H \oplus H^{'})/(Q \oplus H^{'}) \cong H/Q$ is a subdirectly irreducible module.

Now, $G \subseteq_e E = H \oplus H^{'}$ and $P = Q \oplus H^{'} \subset_e H \oplus H^{'}$. Therefore, $G \subseteq_e P$. Hence, by our assumption, $P = \oplus_{k\in\mathcal{K}} W_k$, where each W_k is a nonzero injective module. Since $P \subset_e E$ and $P \neq E$, P is not injective and so $|\mathcal{K}| = \infty$.

We claim that for any finite subset $\mathcal{L}$ of $\mathcal{K}$ and for any positive integer n there exists $i > n$ such that $V_i \cap (\oplus_{k\in\mathcal{L}} W_k)$ is not essential in V_i.

Suppose the above claim is not true. Then there exists a finite subset $\mathcal{L} \subseteq \mathcal{K}$ and an integer $n \geq 1$ such that $V_i \cap (\oplus_{k\in\mathcal{L}} W_k) \subset_e V_i$ for all $i > n$. Let A be an essential closure of $\oplus_{i>n}(V_i \cap (\oplus_{k\in\mathcal{L}} W_k))$ in $\oplus_{k\in\mathcal{L}} W_k$ which is injective and so A is also injective.

We have $\oplus_{i\geq n+1}(V_i \cap \oplus_{k\in\mathcal{L}} W_k) \subset_e A \subset \oplus_{k\in\mathcal{L}} W_k$. This gives, $V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus_{i\geq n+1} (V_i \cap \oplus_{k\in\mathcal{L}} W_k) \subset_e V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A \subset E = H \oplus H^{'}$. Therefore, $(V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus_{i\geq n+1} (V_i \cap \oplus_{k\in\mathcal{L}} W_k)) \cap H \subset (V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A) \cap H \subset H$.

Since $V_i \cap (\oplus_{k\in\mathcal{L}} W_k) \subset_e V_i$ for all $i > n$, we have $V_1 \oplus V_2 \oplus ... \oplus V_n \oplus_{i\geq n+1} (V_i \cap \oplus_{k\in\mathcal{L}} W_k) \subset_e \oplus_{i\in\mathbb{N}} V_i \subset_e H$. Setting $B = V_1 \oplus V_2 \oplus ... \oplus V_n \oplus A$, we obtain that $B \cap H = (V_1 \oplus V_2 \oplus ... \oplus V_n \oplus A) \cap H$ is an essential submodule of H. Furthermore, $B \cap H \subset_e B$. For, if $0 \neq b \in B$ then $b = v_1 + ... + v_n + a$ where $v_i \in V_i$, $a \in A$. If $a = 0$, $b \in H$ and we are done. If $a \neq 0$ then there exists $r \in R$ such that $0 \neq ar \in \oplus_{i\geq n+1}(V_i \cap \oplus_{k\in\mathcal{L}} W_k)$ and hence $ar \in \oplus_{i\in\mathbb{N}} V_i$. Thus $0 \neq br \in H$.

We know that $H^{'}$ is a complement of H in E. Now we claim that $H^{'}$ is a complement of B in E as well. To show this we need to show that $H^{'}$ is maximal submodule of E with respect to the property that $H^{'} \cap B = 0$. If $H^{'} \cap B \neq 0$, then since $H \cap B \subset_e B$, we have $(H^{'} \cap B) \cap (H \cap B) \neq 0$. This implies $(H^{'} \cap H) \cap B \neq 0$, a contradiction because $H^{'} \cap H = 0$. Therefore, $H^{'} \cap B = 0$. Now to complete the proof, let $L \supset H^{'}$ such that $L \cap B = 0$. We first show that $L \cap H = 0$. If $L \cap H \neq 0$, then since $L \cap H$ is a nonzero submodule of H and $H \cap B \subset_e H$, we have $(L \cap H) \cap (H \cap B) \neq 0$, which gives $(L \cap B) \cap H \neq 0$, a contradiction. Thus $L \cap H = 0$. Since $H^{'} \subset L$ and $H^{'}$ is a complement of H in E, we get $H^{'} = L$, proving that $H^{'}$ is a complement of B in E.

Therefore, $B \oplus H^{'} \subseteq_e E$. But since both B and $H^{'}$ are injective, $B \oplus H^{'}$ is injective. Thus $E = B \oplus H^{'} = (V_1 \oplus V_2 \oplus ... \oplus V_n \oplus A) \oplus H^{'} \subseteq Q + P + H^{'} = P$, a contradiction because $P \subset E$ and $P \neq E$.

This proves that for any finite subset $\mathcal{L}$ of $\mathcal{K}$ and for any positive integer n there exists $i > n$ such that $V_i \cap (\oplus_{k\in\mathcal{L}} W_k)$ is not essential in V_i.

We now proceed by induction to construct a sequence of submodules $\{W^{'}_{k_j} : j = 1, 2, ..., n, ...\}$ such that each $W^{'}_{k_j}$ is a nonzero injective submodule of W_{k_j} isomorphic to a submodule $V^{'}_{i_j}$ of V_{i_j}, where $k_1, k_2, ..., k_n, ...$ are distinct elements of $\mathcal{K}$ and $1 \leq i_1 < i_2 < ... < i_n < ...$

Let $i_1 \geq 1$ be arbitrary. Now $V_{i_1} \subset \oplus_{k\in\mathcal{K}} W_k$ implies, there exists a nonzero submodule $V^{'}_{i_1}$ of V_{i_1} such that $V^{'}_{i_1}$ is isomorphic to a submodule $W^{'}_{k_1}$ of W_{k_1} for some $k_1 \in \mathcal{K}$. Clearly, we may choose $V^{'}_{i_1}$ to be injective submodule of V_{i_1}.

For $n \geq 1$, assume that we have a sequence $\{W^{'}_{k_j} : j = 1, 2, ..., n\}$ with the above stated property. By the fact proved above, there exists $i_{n+1} > i_n$ such that $X = V_{i_{n+1}} \cap (\oplus_{k\in\mathcal{K}_1} W_k)$ is not essential in $V_{i_{n+1}}$, where $\mathcal{K}_1 = \{k_1, k_2, ..., k_n\}$. Let $X^{'}$ be a complement of X in $V_{i_{n+1}}$. Then $X^{'} \cap (\oplus_{k\in\mathcal{K}_1} W_k) = X^{'} \cap X = 0$. We have $X^{'} \subset V_{i_{n+1}} \subset (\oplus_{k\in\mathcal{K}_1} W_k) \oplus (\oplus_{k\in\mathcal{K}_2} W_k)$, where $\mathcal{K}_2 = \mathcal{K}\backslash\mathcal{K}_1$. Let $\pi : (\oplus_{k\in\mathcal{K}_1} W_k) \oplus (\oplus_{k\in\mathcal{K}_2} W_k) \longrightarrow \oplus_{k\in\mathcal{K}_2} W_k$ be the projection. Then for $\pi|_{X'} : X^{'} \longrightarrow \oplus_{k\in\mathcal{K}_2} W_k$, $ker(\pi|_{X'}) = X^{'} \cap (\oplus_{k\in\mathcal{K}_1} W_k) = 0$. Therefore, $X^{'}$ is isomorphic to some submodule C of $\oplus_{k\in\mathcal{K}_2} W_k$. So, the module C contains a nonzero submodule which is isomorphic to a submodule F of $W_{k_{n+1}}$ for some $k_{n+1} \in \mathcal{K}_2$. Denote by $W^{'}_{k_{n+1}}$ an essential closure of F in $W_{k_{n+1}}$. Since F is isomorphic to a submodule of the injective module $V_{i_{n+1}}$, we conclude that $W^{'}_{k_{n+1}}$ is isomorphic to a submodule of $V_{i_{n+1}}$ as well. Obviously the family $\{W^{'}_{k_j} : j = 1, 2, ..., n+1\}$ satisfies the required property. This completes the induction argument.

Now set $\mathcal{K}^{'} = \{k_1, k_2, ..., k_n, ...\}$. Choose disjoint subsets $\mathcal{K}^{'}_1$ and $\mathcal{K}^{'}_2$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}^{'}_1 \cup \mathcal{K}^{'}_2$ and $\mathcal{K}^{'} \cap \mathcal{K}^{'}_1 = \{k_1, k_3, ..., k_{2n+1}, ...\}$. Clearly, $\mathcal{K}^{'} \cap \mathcal{K}^{'}_2 = \{k_2, k_4, ..., k_{2n}, ...\}$.

Now we claim that either $\oplus_{k\in\mathcal{K}^{'}_1} W_k$ is injective or $\oplus_{k\in\mathcal{K}^{'}_2} W_k$ is injective.

Set $V = \oplus_{k\in\mathcal{K}_1'} W_k$ and $W = \oplus_{k\in\mathcal{K}_2'} W_k$. We have $P = V \oplus W$. Let $\widehat{V}$ and $\widehat{W}$ be essential closures of V and W respectively in E. Clearly, both $\widehat{V}$ and $\widehat{W}$ are injective. Now, $P = V \oplus W \subset_e \widehat{V} \oplus \widehat{W} \subset E$. Because $P \subset_e E$, we obtain $E = \widehat{V} \oplus \widehat{W}$. Therefore, $E/P = (\widehat{V} \oplus \widehat{W})/(V \oplus W) \cong (\widehat{V}/V) \times (\widehat{W}/W)$. Since E/P is shown to be subdirectly irreducible in the beginning of the proof, we have either $V = \widehat{V}$ or $W = \widehat{W}$. This proves our claim.

Thus, we may assume, without loss of generality, that the module $\oplus_{k\in\mathcal{K}_1'} W_k$ is injective. Since $\oplus_{n=0}^{\infty} W'_{k_{2n+1}}$ is a direct summand of $\oplus_{k\in\mathcal{K}_1'} W_k$, we get that $\oplus_{n=0}^{\infty} W'_{k_{2n+1}}$ is injective. Recalling that $\oplus_{n=0}^{\infty} V'_{i_{2n+1}} \cong \oplus_{n=0}^{\infty} W'_{k_{2n+1}}$, we conclude that $\oplus_{n=0}^{\infty} V'_{i_{2n+1}}$ is an injective module. This completes the proof.

(b) Assume to the contrary that there exists a cyclic right R-module C with $G\dim_M(C) = \infty$. Then C has an independent family $\{C_i : i = 1, 2, ...\}$ of nonzero submodules such that each C_i is isomorphic to a submodule of M. Set D_i to be closure of C_i in M. Then $\{D_i : i = 1, 2, ...\}$ is a family of injective submodules of M. Therefore by (a), there exists an infinite subset $\mathcal{J} \subseteq \{1, 2, ...\}$ and nonzero injective submodules $D'_j \subseteq D_j$, $j \in \mathcal{J}$, such that $\oplus_{j\in\mathcal{J}} D'_j$ is injective. Set $C'_j = C_j \cap D'_j$, $j \in \mathcal{J}$ and note that $C'_j \neq 0$. Since $\oplus_{j\in\mathcal{J}} D'_j$ is injective, the natural inclusion $C'_j \longrightarrow \oplus_{j\in\mathcal{J}} D'_j$ can be extended to a homomorphism $f : C \longrightarrow \oplus_{j\in\mathcal{J}} D'_j$. Because C is cyclic, there exists a finite subset $\mathcal{K} \subseteq \mathcal{J}$ such that $f(C) \subseteq \oplus_{k\in\mathcal{K}} D'_k$, and so $C'_j = f(C'_j) \subseteq f(C) \cap D'_j = 0$ for all $j \notin \mathcal{K}$. Therefore, $C_j \cap D'_j = 0$ for all $j \notin \mathcal{K}$, which contradicts that $C_i \subset_e D_i$ for each i. Therefore, R is right $q.f.d.$ relative to M. □

3. PROOF OF THEOREM 2

Proof. (b) $\Longrightarrow$ (a). Suppose that $M^{(\lambda)}$ is not injective for some infinite cardinal λ. Set $E = E(M^{(\lambda)})$, pick $x \in E \backslash M^{(\lambda)}$ and let $L = xR$. By Lemma 3 (b), R is right $q.f.d.$ relative to M. From this it follows that every nonzero cyclic and hence every nonzero submodule of M contains a uniform submodule. Now, consider the set $\mathcal{S}$ of independent families $(M_k)_{k\in\mathcal{K}}$ of uniform injective modules $0 \neq M_k \subseteq M$. Suppose $\mathcal{S}$ is partially ordered by $(M_k)_{k\in\mathcal{K}} \leq (N_l)_{l\in\mathcal{L}}$ if and only if $\mathcal{K} \subseteq \mathcal{L}$ and $M_k = N_k$ for $k \in \mathcal{K}$. By Zorn's lemma we get a maximal independent family $(M_i)_{i\in\mathcal{I}}$ of uniform injective submodules. Clearly $\oplus_{i\in\mathcal{I}} M_i \subseteq_e M$, because otherwise we will get a contradiction to the maximality of this independent family of submodules. This yields that we have an independent family $\{W_i \mid i \in \mathcal{I}\}$ of uniform injective submodules of $M^{(\lambda)}$ such that each W_i is isomorphic to a submodule of M and $\oplus_{i\in\mathcal{I}} W_i \subseteq_e M^{(\lambda)}$.

Now we proceed to show that there is a sequence of pairwise distinct elements $i_1, i_2, ...$ in $\mathcal{I}$ and an independent family of direct summands $V_1, V_2, ...$ of E such that $V_j \cong W_{i_j}$ and $\pi_{j-1}(L) \cap V_j \neq 0$ for all $j \geq 1$, where $\mathcal{I}_0 = \mathcal{I}$, and $\mathcal{I}_j = \mathcal{I}_{j-1} \backslash \{i_j\}$ for $i_j \in \mathcal{I}$. Set $E_0 = E$, E_j as an essential closure of $\oplus_{i\in\mathcal{I}_j} W_i$ in E_{j-1}, $\pi_0 = id_E$, and π_j as the projection of E onto E_j along $V_1 \oplus ... \oplus V_j$.

Since $\oplus_{i\in\mathcal{I}} W_i \subseteq_e M^{(\lambda)} \subset_e E$ and L is a nonzero submodule of E, we have $L \cap (\oplus_{i\in\mathcal{I}} W_i) \neq 0$. This implies L contains a nonzero submodule X_1 isomorphic to a submodule of some W_i, say, W_{i_1}. Then, $E(X_1) \cong W_{i_1}$. Set $V_1 = E(X_1)$. Clearly, $L \cap V_1 \neq 0$.

For $n \geq 1$, assume that we have a sequence $\{V_j\}$, $1 \leq j \leq n$, of submodules of M with the above stated properties. Since $x \notin M^{(\lambda)}$, $L = xR \nsubseteq \oplus_{i=1}^{n} V_i = \ker(\pi_n)$, and so $\pi_n(L) \neq 0$. Now $\oplus_{i \in \mathcal{I}_n} W_i \subset_e E_n$ and because $\pi_n : E \longrightarrow E_n$, we have $\pi_n(L) \cap (\oplus_{i \in \mathcal{I}_n} W_i) \neq 0$ Now $\pi_n(L) \cap (\oplus_{i \in \mathcal{I}_n} W_i)$ contains a nonzero cyclic uniform submodule, say, C. This implies, there exists a finite subset $\mathcal{K} = \{k_1, k_2, ..., k_m\} \subseteq \mathcal{I}_n$ such that $C \subseteq \oplus_{k \in \mathcal{K}} W_k$. Let V_{n+1} be the essential closure of C in $\oplus_{k \in \mathcal{K}} W_k$. Since $\oplus_{k \in \mathcal{K}} W_k$ is injective, V_{n+1} is injective. So, $\oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus D$ for some submodule D of $\oplus_{k \in \mathcal{K}} W_k$. Since V_{n+1} is injective, it has the exchange property. Therefore, $\oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K}} W_k^{'})$ for some submodules $W_k^{'}$ of W_k. Since $W_k^{'}$ are injective and each W_k is indecomposable, either $W_k^{'} = 0$ or $W_k^{'} = W_k$. We recall that V_{n+1} is uniform because it is the closure of uniform module C. Comparing the Goldie dimension on each side of $\oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K}} W_k^{'})$, we get that there exists exactly one index $k_t \in \mathcal{K}$ such that $W_{k_t}^{'} = 0$, and for all $k (\neq k_t) \in \mathcal{K}$, $W_k^{'} = W_k$. So, $\oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K} \setminus \{k_t\}} W_k)$. This yields $V_{n+1} \cong (\oplus_{k \in \mathcal{K}} W_k)/(\oplus_{k \in \mathcal{K} \setminus \{k_t\}} W_k) \cong W_{k_t}$. Setting $i_{n+1} = k_t$, we have $V_{n+1} \cong W_{i_{n+1}}$. Note that $\pi_n(L \cap V_{n+1}) \subseteq \pi_n(L) \cap \pi_n(V_{n+1})$. Since π_n is identity on E_n and $V_{n+1} \subset E_n$, $\pi_n(L) \cap \pi_n(V_{n+1}) = \pi_n(L) \cap V_{n+1}$. Also, as $\ker(\pi_n) = V_1 \oplus ... \oplus V_n$, $\pi_n(L \cap V_{n+1}) \neq 0$. Therefore, $\pi_n(L) \cap V_{n+1} \neq 0$. Thus, we have obtained a sequence of submodules $\{V_j\}$, $j = 1, 2, ..., n+1$, with the required properties. This completes the induction argument.

Now we claim that there exists a properly ascending chain $N_0 \subset N_1 \subset ... \subset N_j \subset ...$ of submodules of L such that $N_0 = 0$ and $E(N_j/N_{j-1}) \cong V_j$ for all $j \geq 1$.

Set $N_1 = L \cap V_1$. Since V_1 is a uniform injective module, $E(N_1/N_0) \cong V_1$. By construction of the family $\{V_j\}$, $\pi_1(L) \cap V_2 \neq 0$ and since π_1 is onto, there exists a uniquely determined submodule N_2 of L such that $\pi_1(L) \cap V_2 = \pi_1(N_2)$. Since $\pi_1(N_1) = 0$ but $\pi_1(N_2) \neq 0$, we obtain $N_2 \supset N_1$. Next, by isomorphism theorem, $\pi_1(N_2) \cong N_2/N_2 \cap V_1$. Now $N_2 \cap V_1 = N_2 \cap V_1 \cap L = N_2 \cap N_1 = N_1$. So, $N_2/N_1 \cong \pi_1(N_2) = \pi_1(L) \cap V_2$ and hence $E(N_2/N_1) \cong V_2$. Because π_2 is onto, there exists a uniquely determined submodule N_3 of L such that $\pi_2(N_3) = \pi_2(L) \cap V_3$. Note that $\pi_2 \pi_1 = \pi_2$. Since $\ker(\pi_2) = V_1 \oplus V_2$, $\pi_2(N_2) = \pi_2(\pi_1(N_2)) = \pi_2(\pi_1(L) \cap V_2) = 0$ but $\pi_2(N_3) \neq 0$, we obtain $N_3 \supset N_2$. Now, $\pi_2(N_3) = \pi_2(\pi_1(N_3)) \cong \frac{\pi_1(N_3)}{\pi_1(N_3) \cap (V_1 \oplus V_2)}$. The natural map $\varphi : \pi_1(N_3) \longrightarrow N_3/N_1$ given by $\pi_1(n_3) \longmapsto n_3 + N_1$ for $n_3 \in N_3$ is well-defined because if $n_3 \in \ker(\pi_1|_{N_3}) = V_1 \cap N_3 = N_1$, then $n_3 \in N_1$. This is clearly an isomorphism. Furthermore, the restriction of φ to $\pi_1(N_3) \cap (V_1 \oplus V_2)$ is isomorphism onto N_2/N_1. For, if $\pi_1(n_3) \in V_1 \oplus V_2$ then $\pi_1(n_3) = v_1 + v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$. This gives $\pi_1(n_3) = \pi_1(v_2) = v_2$ and hence $\pi_1(n_3) \in \pi_1(L) \cap V_2 = \pi_1(N_2)$. So, the restriction of φ to $\pi_1(N_3) \cap (V_1 \oplus V_2)$ sends $\pi_1(n_3) \longmapsto n_2 + N_1$, $n_3 \in N_3$, $n_2 \in N_2$ where $\pi_1(n_2) = \pi_1(n_3)$. Clearly, this is also well-defined. For, if $\pi_1(n_3) = 0$, then $\pi_1(n_2) = 0$ and this gives $n_2 \in V_1 \cap N_2 = N_1$. Therefore, $\pi_2(N_3) \cong \frac{\pi_1(N_3)}{\pi_1(N_3) \cap (V_1 \oplus V_2)} \cong \frac{N_3/N_1}{N_2/N_1} \cong N_3/N_2$. Now, $N_3/N_2 \cong \pi_2(N_3) = \pi_2(L) \cap V_3$ and hence $E(N_3/N_2) \cong V_3$.

Continuing in this fashion, we create a properly ascending chain $N_0 \subset N_1 \subset ... \subset N_j \subset ...$ of submodules of L such that $N_0 = 0$ and $E(N_j/N_{j-1}) \cong V_j$ for all $j \geq 1$.

Since $\{V_j\}$, $j \in \mathbb{N}$ is an independent family of uniform injective modules isomorphic to a submodule of M, by above lemma, there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ such that $\oplus_{j\in\mathcal{J}} V_j$ and hence $\oplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ is injective. Set $N = \cup_{i\in\mathbb{N}} N_i$. Given $j \in \mathcal{J}$, let $\alpha_j : N \longrightarrow E(N_j/N_{j-1})$ be the canonical mapping. Let $\alpha : N \longrightarrow \oplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ be defined by $\alpha(x) = \{\alpha_j(x)\}_{j\in\mathcal{J}}$ for all $x \in N$. Since $\oplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ is injective, we may extend α to $\alpha^* : L \longrightarrow \oplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$. As L is finitely generated, there exists a finite subset $\mathcal{K} \subseteq \mathcal{J}$ such that $\alpha^*(L) \subseteq \oplus_{k\in\mathcal{K}} E(N_k/N_{k-1})$. For $j \in \mathcal{J}\backslash\mathcal{K}$ and $x \in N_j$ we have $\alpha_j(x) = x + N_{j-1} = 0$, showing that $N_{j-1} = N_j$, a contradiction.

Therefore, $M^{(\lambda)}$ is injective for any cardinal λ and hence M is Σ-injective.

(a) $\Longrightarrow$ (b) is obvious.

This completes the proof of Theorem 2.

□

As a consequence of Theorem 2, we have the following characterization for a right noetherian ring.

Theorem 4. *Let R be a ring. Then the following are equivalent:*

(i) R is right noetherian.

(ii) For each injective module M_R, there exists an infinite cardinal α such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules.

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (i) follows from Theorem 2 and by Faith-Walker [5] that a ring R is right noetherian if and only if every injective right R-module is Σ-injective. □

Remark 5. *The above result generalizes a result of Beidar-Ke [2] which states that a ring R is right noetherian if and only if every essential extension of a direct sum of injective right R-modules is again a direct sum of injective right R-modules. Note that [2] indeed generalizes a result of Bass [1] that a ring is right noetherian if and only if every direct sum of injective modules is injective.*

Acknowledgment. We would like to thank the referees for several helpful suggestions and comments.

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