GENERALIZED GROUP ALGEBRAS OF LOCALLY COMPACT GROUPS

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Dedicated to the memory of late Professor Irving Kaplansky

Abstract. This paper studies the homological properties of generalized group algebra $L^1(G, A)$ of a locally compact group G over a Banach algebra A with an identity of norm 1. It is shown that if $L^1(G, A)$ is right continuous then G is finite and A is right continuous. It is also shown that $L^1(G, A)$ is right self-injective if and only if G is finite and A is right self-injective.

1. Preliminaries

A module M_R is called N-injective if every R-homomorphism from a submodule L of N to M can be extended to an R-homomorphism from N to M. A module M_R is called quasi-injective or self-injective if it is M-injective. If R_R is quasi-injective then R is called a right self-injective ring.

A lattice L is said to be upper continuous if L is complete and $a \wedge (\vee b_i) = \vee (a \wedge b_i)$ for all $a \in L$ and all linearly ordered subsets $\{b_i\} \subseteq L$. A ring R is called von Neuman regular if for each $a \in R$ there exists an $x \in R$ such that $axa = a$. Von-Neumann called a regular ring R to be right continuous if the lattice $L(R_R)$ of principal right ideals of R is upper continuous, equivalently, for any two right ideals A and B with $A \cap B = 0$, the projection mapping $A \oplus B \longrightarrow A$ can be lifted to an endomorphism of R. It is straightforward that any continuous regular ring satisfies (i) every right ideal is essential in a direct summand, and (ii) every right ideal isomorphic to a summand is itself a summand. In general, a ring R is called right continuous if it satisfies the conditions (i) and (ii). More generally, a module M_R is called continuous if it satisfies the following two conditions: (i) every submodule of M is essential in a direct summand of M, (ii) If a submodule N of M is isomorphic to a direct summand of M then N itself is a direct summand of M. Every right self-injective ring is right continuous but not conversely.

Let R be any ring, not necessarily with identity. Let $J(R)$ be its Jacobson radical. The right singular ideal of R, denoted by $Z(R_R)$, is defined as: $Z(R_R) = \{r \in R : rE = 0$ for some essential right ideal E of R$\}$.

If A is a Banach algebra, then for $x \in A$, $r(x)$ denotes the spectral radius of x.

A topological group is a group G together with a topology such that the maps $G \times G \longrightarrow G$ where $(\alpha, \beta) \mapsto \alpha\beta$ and $G \longrightarrow G$ where $\alpha \mapsto \alpha^{-1}$ are continuous. A topological group G is called a locally compact group if it is Hausdorff and locally

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compact as a topological space. It is well-known that every locally compact group has a left Haar measure unique upto a scalar multiple.

Definition 1. *Let G be a locally compact group with the left Haar measure m. The group algebra $L^1(G)$ is defined as the Banach algebra consisting of all complex-valued m-integrable functions on G, with the norm given as*

$$||\varphi|| = \int_G |\varphi(t)| \; dm(t) \qquad (\varphi \in L^1(G)),$$

and equipped with the convolution product $*$, where

$$(\varphi * \psi)(t) = \int_G \varphi(s)\psi(s^{-1}t) \; dm(s) \qquad (\varphi, \psi \in L^1(G), t \in G).$$

We know that $L^1(G)$ has an approximate identity bounded by 1.

More generally, Hausner defined generalized group algebras of vector-valued integrable functions as below.

Definition 2. *Let A be a Banach algebra with identity of norm 1 and let G be a locally compact group with the left Haar measure m. The generalized group algebra $L^1(G, A)$ is defined as the Banach algebra of all A-valued Bochner integrable functions on G, with the norm given as*

$$||\varphi||_1 = \int_G ||\varphi(t)|| \; dm(t) \qquad (\varphi \in L^1(G, A)),$$

and equipped with the convolution product $*$, where

$$(\varphi * \psi)(t) = \int_G \varphi(s)\psi(s^{-1}t) \; dm(s) \qquad (\varphi, \psi \in L^1(G, A), t \in G).$$

$L^1(G, A)$ can also be thought of as the projective tensor product $L^1(G)\widehat{\otimes}A$, the completion of the algebraic tensor product $L^1(G) \otimes A$ equipped with the projective tensor-norm (see [8] for details). $L^1(G, A)$ is a Banach algebra with an approximate identity bounded by 1.

2. Results

We start by stating some well-known results that play key role in proving our main theorem.

Proposition 3. *(Kaplansky [7]) A von Neumann regular Banach algebra must be finite-dimensional.*

Proposition 4. *(Jacobson [4]) The radical $J(R)$ of a normed ring R is a generalized nil ideal, i.e. if $x \in J(R)$ then $r(x) = \lim_{n\to\infty} ||x^n||^{1/n} = 0$. Also, $J(R)$ is a closed ideal of R.*

Proposition 5. *[9] Let M_R be a continuous module, and let $S = Hom_R(M, M)$. Then $S/J(S)$ is a von Neumann regular ring.*

The proof of this proposition is given in the literature for rings with identity but it can be adapted for rings without identity.

Lemma 6. *(Johnson [6]) Let R be a Banach algebra with an approximate identity bounded by 1. Let T belong to $S(R) = Hom_R(R, R)$. Then T is linear and continuous. Further, $S(R)$ can be made into a Banach algebra with identity, the norm being the usual operator norm.*

Theorem 7. *([1], [11]) Let R be a ring with identity and G be a group. Then RG is right self-injective if and only if R is right self-injective and G is finite.*

The study of group algebras RG of any group G over a ring R that are continuous, quasi-continuous, or more generally CS has been limited to the cases when R is a field. There are almost no results in the literature on the properties of the ring R when RG is continuous or quasi-continuous. Before studying generalized group algebras of locally compact groups, we first consider classical group algebras RG and show that R is continuous (or quasi-continuous) when RG is continuous (or quasi-continuous).

Lemma 8. *Let R be a ring with identity and G be a group. If RG is quasi-continuous (π-injective) then R is right quasi-continuous.*

Proof. Let $\varphi : I_1 \oplus I_2 \longrightarrow I_1$ be an idempotent R-homomorphism where I_1 and I_2 are right ideals of R with $I_1 \cap I_2 = 0$. Define $\bar{\varphi} : (I_1 \oplus I_2)G \longrightarrow I_1G$ by $\bar{\varphi}(\Sigma(a_g + b_g)g) = \Sigma\varphi(a_g)g$. Since RG is quasi-continuous, $\bar{\varphi}$ extends to an endomorphism of RG. So, $\bar{\varphi}(x) = yx$ for some $y \in RG$. Now, if $t \in I_1 \oplus I_2$, then we have $\varphi(t) = \bar{\varphi}(t) = yt$. Let $y = y_0g_0 + y_1g_1 + ... + y_ng_n$ where g_0 is identity of G. This gives, $\varphi(t) = y_0t$ where $y_0 \in R$. Therefore, R is right quasi-continuous. □

Lemma 9. *If R is a quasi-continuous ring such that $Z(R) \subseteq J(R)$, then R is right continuous.*

Proof. The proof given in the literature (e.g. see [9]) assumes $Z(R) = J(R)$. However, simple examination shows that it is enough to assume $Z(R) \subseteq J(R)$. □

Proposition 10. *Let R be a ring with identity and G be a group. If RG is continuous then R is right continuous.*

Proof. By Lemma 8, R is quasi-continuous. To prove that R is continuous, we only need to show that $Z(R) \subseteq J(R)$. Let $a \in Z(R)$. Since RG is continuous, $Z(RG) = J(RG)$. We have $Z(R) \subset Z(R)G \subseteq Z(RG) = J(RG)$. Therefore, $a \in J(RG)$. So, $x = (1 - a)$ is invertible in RG. Hence there exists $y \in RG$ such that $xy = 1 = yx$. Let $y = y_0g_0 + y_1g_1 + ... + y_ng_n$ where g_0 is identity of G. Then, we get $xy_0 = 1$ and $xy_i = 0$ for each $i \geq 1$. Similarly, $y_0x = 1$ and $y_ix = 0$ for each $i \geq 1$. Now, for each $i \geq 1$, $y_0xy_i = 0$ which gives $y_i = 0$ for each $i \geq 1$. Hence $y \in R$. Therefore, $(1 - a)$ is invertible in R. So, $a \in J(R)$. Thus, $Z(R) \subseteq J(R)$. This proves that R is right continuous. □

We are now ready to study continuous generalized group algebras.

Let G be a locally compact group with the left Haar measure m and let A be a Banach algebra with identity of norm 1. Let $M(G)$ denote the measure algebra of G with adjoint operation $\tilde{}$ given by $\widetilde{\mu}(E) = \overline{\mu(E^{-1})}$ for $\mu \in M(G)$ and E measurable with E^{-1} measurable in G. For $\mu \neq 0$, we have $r(\widetilde{\mu} * \mu) \neq 0$.

Theorem 11. *If $L^1(G, A)$ is right continuous then G is finite and A is right continuous.*

Proof. Let $R = L^1(G, A) = L^1(G)\widehat{\otimes}A$ be right continuous. Set $S(R) = Hom_R(R, R)$. By Proposition 5, $S(R)/J(S(R))$ is von Neumann regular. By Lemma 6, every member of $S(R)$ is bounded. So $S(R)$ can be considered as a Banach subalgebra of the algebra of bounded operators on R. Hence, $S(R)/J(S(R))$ is a Banach algebra. So by Kaplansky (Proposition 3), $S(R)/J(S(R))$ is finite-dimensional.

Now we claim that $M(G)$ is embeddable in $S(R)/J(S(R))$ as an algebra.

For every $\nu \in M(G)$, consider the map $W_\nu = L_\nu \otimes id_A \in S(R)$, where $L_\nu(f) = \nu * f$, $f \in L^1(G)$. Then the map $W : M(G) \longrightarrow S(R)$ given by $\nu \longmapsto W_\nu$ is a norm-preserving isomorphism onto the Banach subalgebra $W(M(G))$. Let $\mu(\neq 0) \in M(G)$. Then, since $W_\mu(f \otimes a) = (\mu * f) \otimes a$, $||W_\mu|| = ||\mu||$. Also, $||W_\mu^n|| = ||\mu^n||$. As a consequence, $r(W_\mu) = r(\mu)$. Thus, $r(W_{\widetilde{\mu}*\mu}) = r(\widetilde{\mu} * \mu) \neq 0$.

We claim $W_\mu \notin J(S(R))$. If possible, let $W_\mu \in J(S(R))$. Then $W_{\widetilde{\mu}}W_\mu \in J(S(R))$. This gives $W_{\widetilde{\mu}*\mu} \in J(S(R))$. Hence by Proposition 4, $r(W_{\widetilde{\mu}*\mu}) = 0$, a contradiction. Thus, $W_\mu \notin J(S(R))$ as claimed.

Let π be the canonical homomorphism from $S(R)$ to $S(R)/J(S(R))$. Then the composition $\pi W : M(G) \xrightarrow{W} S(R) \xrightarrow{\pi} S(R)/J(S(R))$ is a one-to-one homomorphism and so $M(G)$ embeds in $S(R)/J(S(R))$ as an algebra.

Thus, $M(G)$ is finite-dimensional. Hence, G is finite. Therefore, $L^1(G, A) = AG$. Then, by Proposition 10, A is right continuous. □

Note that since $L^1(G)$ is an algebra with involution, it has left-right symmetry.

Corollary 12. *$L^1(G)$ is continuous if and only if G is finite. In this case $\mathbb{C}G = L^1(G)$.*

Remark 13. *It is known that for any field K if KG is continuous then G is locally finite but the converse need not be true. For examples of infinite locally finite groups G such that KG is continuous, we refer the reader to [5].*

Theorem 14. *$L^1(G, A)$ is right self-injective if and only if G is finite and A is right self-injective.*

Proof. Let $R = L^1(G, A)$ be right self-injective. Then by Theorem 11, G is finite. As a consequence, $R = A[G]$. Therefore, A is right self-injective. Conversely, if G is finite and A is right self-injective then $L^1(G, A) = AG$ is right self-injective. □

Corollary 15. *$L^1(G)$ is self-injective if and only if G is finite.*

References

1. I. G. Connell, On the Group Ring, Can. J. Math. 15 (1963), 650-685.

2. C. Faith, Y. Utumi, Quasi-injective modules and their endomorphism rings, Arch. der Math., 15 (1964), 166-174.

3. A. Hausner, On Generalized Group Algebras, Proc. Amer. Math. Soc. 10 (1959), 1-10.

4. N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math., 67 (1945), 300-320.

5. S. K. Jain, P. Kanwar and J. B. Srivastava, When is a Semilocal Group Algebra Continuous?, To appear in Journal of Algebra, available online, March 2007.

6. B. E. Johnson, Continuity of centralisers on Banach algebras, J. London Math. Soc. 41 (1966), 639-640.

7. I. Kaplansky, Regular Banach algebras, J. Indian Math. Soc. (N.S.) 12 (1948), 57–62.

8. K. B. Laursen, Ideal Structure in Generalized Group Algebras, Pac. J. Math. 30, 1 (1969), 155-174.

9. S. H. Mohamed and B. J. Muller, Continuous and Discrete Modules, Cambridge University Press, 1990.

10. M. A. Naimark, Normed Rings, P. Noordhoff N. V. - Groningen, 1964.

11. G. Renault, Sur les anneaux de groupes, C. R. Acad. Sci. Paris Sér. A-B 273 (1971), 84–87.

12. I. E. Segal, The Group Algebra of a Locally Compact Group, Trans. Amer. Math. Soc. 61, (1947), 69-105.

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