Essential Extensions of a Direct Sum of Simple Modules

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Dedicated to Don Passman on his 65 th Birthday

Abstract. It is known that every essential extension of a direct sum of injective hulls of simple R-modules is a direct sum of injective R-modules if and only if the ring R is right noetherian. The purpose of this paper is to study the rings R having the property: every essential extension of a direct sum of simple R-modules is a direct sum of quasi-injective R-modules. A commutative ring with this property is known to be an artinian principal ideal ring. In the present paper, we show that such a ring is directly finite. For a right nonsingular ring R with this property, we show that the maximal right ring of quotients $Q^r_{\max}(R)$ is the direct product of a finite number of matrix rings over abelian regular self-injective rings. For a von-Neumann regular ring R, we show that R is noetherian if and only if every essential extension of a direct sum of simple R-modules is a quasi-injective R-module.

1. INTRODUCTION

This paper is inspired by several results that characterize right noetherian rings in terms of direct sums of injective modules as obtained by Bass [1], Papp [19], Kurshan [14], Goursaud-Valette [11], Beidar-Ke [2], Beidar-Jain [3], and other authors (see e.g. [5], [16] and [20]). Also, there are several characterizations of right noetherian rings in terms of decomposition of injective modules due to Matlis [17], Papp [19] and Faith-Walker [9]. For the sake of providing the reader some background, we state here key results relevant to the present study that characterize noetherian rings in terms of direct sums of injective modules.

Theorem 1. *(see [16]) Let R be a ring. Then the following statements are equivalent:*
(1) R is right noetherian;
(2) Every direct sum of injective right R-modules is injective;
(3) Every countable direct sum of injective hulls of simple right R-modules is injective.

Beidar and Ke [2] obtained the following generalization of the above stated theorem

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Theorem 2. *Let R be a ring. Then the following statements are equivalent:*

(1) R is right noetherian;

(2) Every essential extension of a direct sum of a family of injective right R-modules is a direct sum of injective modules;

(3) Given a family $\{S_i : i = 1, 2, ...\}$ of simple right R-modules, each essential extension of $\oplus_{i=1}^{\infty} E(S_i)$ is a direct sum of injective modules;

(4) For any family $\{S_i : i = 1, 2, ...\}$ of simple right R-modules, there exists an infinite subset I of natural numbers such that $\oplus_{i\in I} E(S_i)$ is an injective module.

Later, Beidar and Jain [3] proved that a ring R is right noetherian if and only if R is right $q.f.d.$ and every essential extension of a direct sum of injective hulls of simple modules is a direct sum of quasi-injective modules. However, in the case of a commutative ring, using arguments based on ultrafilters they proved that every essential extension of a semisimple R-module is a direct sum of quasi-injective modules if and only if R is an artinian principal ideal ring. It may be recalled here that Kurshan [14] proved that a ring R is right noetherian if and only if R is right $q.f.d.$ and has the property that every submodule of a cyclic R-module with simple essential socle is finitely generated (Kurshan called such rings TC rings). Faith proved that a right R-module M is noetherian if and only if M is $q.f.d.$ and satisfies the *acc* on subdirectly irreducible (or colocal) submodules ([6], [7]).

In the present paper, we study the class of rings R which satisfy the condition:

(*) Every essential extension of a direct sum of simple R-modules is a direct sum of quasi-injective R-modules.

Throughout this paper, this condition will be referred to as condition (*).

We show that if a ring R satisfies (*) then R is directly finite (Theorem 4). For a right nonsingular ring R, we show that if R satisfies (*) then the maximal right ring of quotients $Q^r_{\max}(R)$ is the direct product of a finite number of matrix rings over abelian regular self-injective rings (Theorem 6). In case of a von-Neumann regular ring R, we show that R is noetherian if and only if R satisfies the condition that every essential extension of a direct sum of simple R-modules is a quasi-injective module (Theorem 8).

2. DEFINITIONS AND NOTATIONS

All rings considered in this paper have unity and all modules are right unital. Let M be an R-module. We denote by $Soc(M)$ and $E(M)$, respectively, the socle and the injective hull of M. We shall write $N \subseteq_e M$ whenever N is an essential submodule of M. A module M is called N-injective, if every R-homomorphism from a submodule L of N to M can be lifted to a R-homomorphism from N to M. A module M is said to be quasi-injective if it is M-injective. Readers may refer to [8], [16] and [18] for basic properties of quasi-injective modules. A module M is called directly finite (or "von-Neumann finite" or "Dedekind finite") if M is not isomorphic to any proper direct summand of itself. A ring R is called directly finite if R is directly finite as an R-module, equivalently, $xy = 1$ implies $yx = 1$, for all $x, y \in R$. A ring R is called von-Neumann regular if every principal right (left) ideal of R is generated by an idempotent. A regular ring is called abelian if all its idempotents are central. An idempotent e in a regular ring R is called abelian idempotent if the ring $e\ R\ e$ is abelian. An idempotent e in a regular right self-injective ring is called faithful idempotent if 0 is the only central idempotent orthogonal to e, that is, $ef = 0$ implies $f = 0$, where f is a central idempotent. A regular right self-injective ring is said to be of Type I provided it contains a faithful abelian idempotent. A regular right self-injective ring R is said to be of Type II provided R contains a faithful directly finite idempotent but R contains no nonzero abelian idempotents. A regular right self-injective ring is of Type III if it contains no nonzero directly finite idempotents. A regular right self-injective ring is of (i) Type I_f if R is of Type I and is directly finite, (ii) Type I_∞ if R is of Type I and is purely infinite, (iii) Type II_f if R is of Type II and is directly finite, (iv) Type II_∞ if R is of Type II and is purely infinite (see [10], pp. 111-115). If R is a regular right self-injective ring of Type I_f then $R \simeq \Pi R_n$ where each R_n is an $n \times n$ matrix ring over an abelian regular self-injective ring (see [10], p.120). The index of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The index of a two-sided ideal J in R is the supremum of the indices of all nilpotent elements of J. If this supremum is finite, then J is said to have bounded index. A ring R is said to be right $q.f.d.$ if every cyclic right R-module has finite uniform (Goldie) dimension, that is, every direct sum of submodules of a cyclic module has finite number of terms. We shall say that Goldie dimension of N with respect to U, $G\dim_U(N)$, is less than or equal to n, if for any independent family $\{V_j : j \in \mathcal{J}\}$ of nonzero submodules of N such that each V_j is isomorphic to a submodule of U, we have that $|\mathcal{J}| \leq n$. Next, $G\dim_U(N) < \infty$ means that $G\dim_U(N) \leq n$ for some positive integer n. The module N is said to be $q.f.d.$ relative to U if for any factor module $\bar{N}$ of N, $G\dim_U(\bar{N}) < \infty$. Note that if $V \subseteq_e U$, then $G\dim_U(N) = G\dim_V(N)$ for all N. If U is the direct sum of cyclic modules, then $G\dim_U(N) = G\dim(N)$. Given two R-modules M and N, we set $Tr_M(N) = \sum\{f(M) : f \in Hom(M_R, N_R)\}$. A ring R is called a q-ring if every right ideal of R is quasi-injective (see [13], [4]). The Jacobson radical of a ring R is denoted by $J(R)$. The maximal right ring of quotients of ring R is denoted by $Q^r_{\max}(R)$.

3. MAIN RESULTS

We begin with a key lemma whose proof is rather technical.

Lemma 3. *Let R be a ring which satisfies the condition (*) and let N be a finitely generated R-module. Then there exists a positive integer n such that for any simple R-module S, we have*

$$G\dim_S(N) \leq n.$$

Proof. If possible, let $G\dim_S(N) = \infty$ for some simple submodule S of N. Let D be a complement of $Tr_S(N)$ in N. Then $Tr_S(N) \oplus D \subset_e N$. Factoring out by D we get that $Tr_S(N)$ is essentially embeddable in N/D. That means $Tr_S(N) \cong B/D$ for some $B/D \subset_e N/D$. This gives that $Tr_{\bar{S}}(N/D) = B/D \subset_e N/D$. Because N/D is also finitely generated, and $G\dim_{\bar{S}}(N/D) = \infty$ we may assume, without any loss of generality, that $Tr_S(N) \subseteq_e N$. Therefore, $Soc(N) = Tr_S(N)$ and $Soc(N) \subset_e N$. Hence, by (*) we get $N = \oplus_{k=1}^{\infty} Q_k$, where each Q_k is quasi-injective. Since N is finitely generated, we conclude that $N = \oplus_{k=1}^{n} Q_k$. Now, $Soc(N) = \oplus_{k=1}^{n} Soc(Q_k)$. Thus, there exists an index $1 \leq k \leq n$ such that $G\dim_S(Soc(Q_k)) = \infty$. Since Q_k is also finitely generated and quasi-injective, we may now, without any loss of generality, further assume that N is quasi-injective. Next, we choose an independent family $\{T_i \mid i \in \mathcal{I}\}$ of submodules of $Soc(N)$ such that $\oplus_{i\in\mathcal{I}} T_i = Soc(N)$. Clearly each T_i is isomorphic to S. Let $\hat{T_i}$ be an essential closure of T_i in N, $i \in \mathcal{I}$. Since $\{T_i | i \in \mathcal{I}\}$ is an independent family of submodules of N, so is $\{\hat{T_i} | i \in \mathcal{I}\}$. Since $|\mathcal{I}|$ is infinite, $\oplus_{i\in\mathcal{I}} \hat{T_i}$ is not finitely generated and so $\oplus_{i\in\mathcal{I}} \hat{T_i} \neq N$. Let L be a maximal submodule of N containing $\oplus_{i\in\mathcal{I}} \hat{T_i}$. Since $Soc(N) \subseteq_e N$ and $Soc(N) = \oplus_{i\in\mathcal{I}} T_i \subset \oplus_{i\in\mathcal{I}} \hat{T_i} \subset L \subset N$, we conclude that $Soc(N)$ is an essential submodule of L and so (*) implies that $L = \oplus_{k\in\mathcal{K}} U_k$ where each U_k is quasi-injective. We claim that $|\mathcal{K}| < \infty$.

Assume to the contrary that $|\mathcal{K}| = \infty$. Choose two infinite disjoint subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Set $V_j = \oplus_{k\in\mathcal{K}_j} U_k$, $j = 1, 2$. Then $L = V_1 \oplus V_2$. For $j = 1, 2$, let W_j be an essential closure of V_j in N. Then each W_j is a direct summand of N and is N-injective since N is quasi-injective. So $W_1 \oplus W_2$ is also a direct summand and N-injective. Now $Soc(N) \subseteq_e W_1 \oplus W_2 \subset_e N$, and therefore $N = W_1 \oplus W_2$. Hence, $N/L = (W_1 \oplus W_2)/(V_1 \oplus V_2) = W_1/V_1 \times W_2/V_2$. Since N/L is a simple module, either $W_1 = V_1$ or $W_2 = V_2$. Since each W_i is a direct summand of N, it is finitely generated. Thus, $W_i \neq V_i$ for $i = 1, 2$, a contradiction. Therefore $|\mathcal{K}| < \infty$.

Let $U_k \subset \hat{U}_k \subset N$, where $\hat{U}_k$ is an essential closure of U_k in N and $k \in \mathcal{K} = \{1, 2, ..., m\}$. Then $U_1 \oplus ... \oplus U_m \subset \hat{U}_1 \oplus ... \oplus \hat{U}_m \subset_e N$. But, since N is quasi-injective, $\hat{U}_1 \oplus ... \oplus \hat{U}_m$ is a direct summand of N. Therefore, $N = \hat{U}_1 \oplus ... \oplus \hat{U}_m$. Then $\frac{N}{L} = \frac{\hat{U}_1 \oplus ... \oplus \hat{U}_m}{U_1 \oplus ... \oplus U_m} \cong \frac{\hat{U}_1}{U_1} \times ... \times \frac{\hat{U}_m}{U_m}$. Since N/L is simple, there exists an index $l \in \mathcal{K}$ such that $\hat{U}_k = U_k$ for all $k \neq l$. If $\hat{U}_l = U_l$, then clearly $L = N$, a contradiction. Therefore $U_l \subset \hat{U}_l$. In particular, U_l is not N-injective.

Next, $N/L = (\oplus_{k \in \mathcal{K}} \hat{U}_k)/(\oplus_{k \in \mathcal{K}} U_k) = \hat{U}_l\ /U_l$ and so $\hat{U}_l\ /U_l$ is a simple module. We claim that $G\dim_S(Soc(U_l)) = \infty$. Else, let $G\dim_S(Soc(U_l)) < \infty$. We now proceed to show that this leads to a contradiction. Let π be the canonical projection of $L = \oplus_{k \in \mathcal{K}} U_k$ onto U_l. Choose independent family of simple submodules $P_1, P_2, ..., P_t$ of U_l such that $P = \oplus_{j=1}^t P_j = Soc(U_l)$. From $Soc(N) \subset_e L$, it follows by intersecting both sides with U_l that $P = \oplus_{j=1}^t P_j = Soc(U_l) \subseteq_e U_l$. Then for every $1 \leq j \leq t$, there exists a finite subset $\mathcal{I}_j \subseteq \mathcal{I}$ such that $P_j \subseteq \oplus_{s \in \mathcal{I}_j} T_s$. Let $\hat{P}_j$ be an essential closure of P_j in $\oplus_{s \in \mathcal{I}_j} \hat{T}_s$. Since $\oplus_{s \in \mathcal{I}_j} \hat{T}_s$ is an N-injective submodule of L, so is $\hat{P}_j$. As $P_j \subseteq U_l$, $\pi|_{P_j}$ is monomorphism and so $\pi(\hat{P}_j) \cong \hat{P}_j$. Recalling that $P = Soc(U_l) \subseteq_e U_l$, we now conclude that $\oplus_{j=1}^t \pi(\hat{P}_j) \subset_e U_l$ and so U_l is N-injective, a contradiction. Therefore, $G\dim_S(Soc(U_l)) = \infty$.

Set $\Lambda = End(\hat{U}_l), \Delta = End(\hat{U}_l/U_l)$ and $\Omega = End(Soc(\hat{U}_l))$. Then $\Lambda U_l = U_l$ and so each element $\lambda \in \Lambda$ induces an endomorphism of the factor module $\hat{U}_l/U_l$. Therefore, there exists a ring homomorphism $f : \Lambda \longrightarrow \Delta$. Set $I = \ker(f)$ and note that $I \neq \Lambda$. Since Δ is a division ring, Λ/I is a domain. Next, $\lambda(Soc(\hat{U}_l)) \subseteq Soc(\hat{U}_l)$ and so the map $g : \Lambda \longrightarrow \Omega$, where $g(\lambda) = \lambda|_{Soc(\hat{U}_l)}$, is a homomorphism of rings. Since $\hat{U}_l$ is quasi-injective, g is a surjective map. Now, $J(\Lambda) = \{\alpha \in \Lambda : \ker(\alpha) \subset_e \hat{U}_l\}$ (see [8], p.44). It can be shown that ker $(g) = J(\Lambda)$. Also, it is known that idempotents modulo $J(\Lambda)$ can be lifted to Λ (see [8], p.48). As shown above in the previous paragraph, $G\dim_S(Soc(\hat{U}_l)) = \infty$. So, $Soc(\hat{U}_l)$ is a direct sum of infinitely many modules each isomorphic to S. Therefore, there exist two isomorphic submodules L_1 and L_2 such that $Soc(\hat{U}_l) = L_1 \oplus L_2$. Let $e_i \in \Omega$ be the canonical projection of $Soc(\hat{U}_l)$ onto L_i, $i = 1, 2$. Clearly, $e_1 e_2 = 0 = e_2 e_1$, $e_1 + e_2 = 1$. Since $\frac{\Lambda}{J(\Lambda)} \cong \Omega$ and g is surjective, there exist $v_i \in \Lambda$ such that $g(v_i) = e_i$ for $i = 1, 2$. This gives $v_i^2 - v_i \in J(\Lambda)$. This implies that there exist $u_i^2 = u_i \in \Lambda$, $i = 1, 2$ such that $u_i - v_i \in J(\Lambda) = \ker(g)$. This gives that $g(u_i) = g(v_i) = e_i$. Since orthogonal idempotents can be lifted to orthogonal idempotents, $u_1 u_2 = 0 = u_2 u_1$. Also, since $e_1 + e_2 = 1$, we have $u_1 + u_2 = 1$. By (Proposition 21.21, [15]), there exist $c \in u_1 \Lambda u_2$ and $d \in u_2 \Lambda u_1$ with $cd = u_1$, $dc = u_2$. Since Λ/I is a domain, either $u_1 \in I$, or $u_2 \in I$. In both cases $c, d \in I$ and so $u_1 = cd \in I$

and $u_2 = dc \in I$ forcing $1 = u_1 + u_2 \in I$, a contradiction. Therefore, $G\dim_S(N)$ can not be infinite. So, for any finitely generated module N and for any simple submodule S we have $G\dim_S(N) < \infty$.

Next, we proceed to show that there exists a positive integer n such that $G\dim_S(N) \leq n$ for each simple module S.

Else, suppose for any integer $n > 1$, there exists a simple submodule $S_n \subseteq N$ such that $G\dim_{S_n}(N) > n$. Then there exists a family of pairwise non-isomorphic simple modules $\{T_i \mid i = 1, 2, ...\}$ such that $G\dim_{T_i}(N) \geq i$. As explained in the beginning of the proof of this lemma we will, without any loss of generality, assume that $Soc(N) \subseteq_e N$. By (*) we know that $N = \oplus_{k=1}^{\infty} Q_k$ where each Q_k is quasi-injective. Since N is finitely generated, we conclude that $N = \oplus_{k=1}^{n} Q_k$. Clearly there exists an index $1 \leq l \leq n$ and an ascending sequence of indexes $i_1, i_2, ..., i_j, ...$ such that $G\dim_{T_{i_j}}(Q_l) \geq j$. Therefore there exists an independent family of simple submodules $\{S_{pq} | p = 1, 2, ..., q = 1, 2, ..., p\}$ of Q_l such that $S_{ij} \cong S_{pq}$ if and only if $p = i$. Let L be an essential closure of $\oplus_{i,j} S_{ij}$ in Q_l. Then L is both a quasi-injective module and a direct summand of Q_l. Hence L is a direct summand of N and so L is finitely generated. Let $\overset{\Lambda}{S}_{ij}$ be an essential closure of S_{ij} in L. Since $S_{ij} \cong S_{ik}$, $\overset{\Lambda}{S}_{ij} \cong \overset{\Lambda}{S}_{ik}$ for all $1 \leq j, k \leq i$ and $i = 1, 2, ...$ We now set $U_k = \oplus_{i=k}^{\infty} \overset{\Lambda}{S}_{ik}$ and denote by W_k an essential closure of U_k in L. Note that W_k is finitely generated. Therefore, $W_k \neq U_k$ for all $k = 1, 2, ...$In particular, there exists a maximal submodule P_1 of W_1 containing U_1. Let $k > 1$ and let $W_k^{'}$ be an essential closure of $\oplus_{i=k}^{\infty} \overset{\Lambda}{S}_{i1}$ in W_1. So, it follows that $W_k^{'} \cong W_k$ for all $k = 2, 3, ...$ Note that $W_1 = (\oplus_{i=1}^{k-1} \overset{\Lambda}{S}_{i1}) \oplus W_k^{'}$. Since $U_1 = \oplus_{i=1}^{\infty} \overset{\Lambda}{S}_{i1} \subset P_1$, we have $\overset{\Lambda}{S}_{i1} \subset P_1$, and so we conclude from modular law that $P_1 = (\oplus_{i=1}^{k-1} \overset{\Lambda}{S}_{i1}) \oplus P_k^{'}$ where $P_k^{'} = P_1 \cap W_k^{'}$. Therefore, $W_1/P_1 \cong W_k^{'}/P_k^{'}$ which implies $P_k^{'}$ is a maximal submodule of $W_k^{'}$ for all $k = 2, 3, ...$Furthermore, as $W_k^{'} \cong W_k$, there exists a maximal submodule P_k of W_k such that $W_k/P_k \cong W_k^{'}/P_k^{'} \cong W_1/P_1$. Set $P = \oplus_{k=1}^{\infty} P_k$ and $S = W_1/P_1$. Now, $\frac{L}{P} \supset \frac{\oplus W_k}{\oplus P_k} \cong \frac{W_1}{P_1} \times \frac{W_2}{P_2} \times ....$This gives that $G\dim_S(L/P) = \infty$, which is not true because, as shown earlier, any finitely generated module has finite Goldie dimension with respect to any simple module. This completes the proof. □

Theorem 4. *Let R be a ring which satisfies the condition (*). Let K, L be R-modules with K finitely generated and $K \subseteq_e L$. Let $\Lambda = End(L)$. Then Λ is directly finite.*

Proof. Assume that Λ is directly infinite. Then, there exist $x, y \in \Lambda$ such that $xy = 1$ and $yx \neq 1$. Set $e_{ij} = y^{i-1}x^{j-1} - y^i x^j$ for all $i, j = 1, 2, ...$It can be easily checked that $\{e_{ij} | i, j = 1, 2, ...\}$ is an infinite set of nonzero matrix units. Let $n > 1$. Since $K \subseteq_e L$, there exists a nonzero cyclic submodule $U_{n,1}$ of K such that $U_{n,1} \subseteq e_{n^2,n^2} L \cap K$. Now we produce cyclic submodules $U_{n,i} \subseteq K$, where $U_{n,i} \cong U_{n,j}$ for all $i, j = 2, 3, ..., n$.

We now produce these cyclic submodules $U_{n,i}$ of K by induction. Consider the module $U_{n,1}$ defined in the previous paragraph. Choose $x_2 \in e_{n^2+1,n^2} U_{n,1} \cap K$. Then $x_2 = e_{n^2+1,n^2} x_1$, where $x_1 \in U_{n,1}$. Denote $U_{n,2} = x_2 R$ and redefine $U_{n,1}$ by setting $U_{n,1} = x_1 R$. Define the module homomorphism $\varphi : U_{n,1} \longrightarrow U_{n,2}$ by $\varphi(x) = e_{n^2+1,n^2} x$. Clearly, this is an epimorphism. Suppose $e_{n^2+1,n^2} x = 0$. Then $e_{n^2,n^2+1}(e_{n^2+1,n^2} x) = 0$. which gives $e_{n^2,n^2} x = 0$ and hence $x = 0$. Therefore φ is an isomorphism, and so $U_{n,1} \cong U_{n,2}$. Suppose now that we have defined cyclic submodules $U_{n,1} \cong U_{n,2} \cong ... \cong U_{n,j-1}$ in K, where $U_{n,i} = x_i R$, $i = 1, 2, ..., j-1$. Next, we choose x_j such that $x_j \in e_{n^2+j-1,n^2+j-2} U_{n,j-1} \cap K$ and write $x_j = e_{n^2+j-1,n^2+j-2} x_{j-1} r_{j-1}$ where $r_{j-1} \in R$. Let $x_{j-1}^{'} = x_{j-1} r_{j-1}$, and set $U_{n,j} = x_j R$. Now redefine $U_{n,j-1} = x_{j-1}^{'} R$ (which is contained in the previously constructed $U_{n,j-1}$) . Then $U_{n,j-1} \cong U_{n,j}$ under the isomorphism that sends

$x \in U_{n,j-1}$ to $e_{n^2+j-1,n^2+j-2}x$. We redefine preceding $U_{n,1}$, $U_{n,2}, ..., U_{n,j-2}$ accordingly so that they all remain isomorphic to each other and to $U_{n,j-1}$. Note that the family $\{U_{n,i} | n = 2, 3, ..., i = 1, 2, ..., n\}$ is independent. By our construction, $U_{n,i} \cong U_{n,j}$ for all $n = 2, 3, ...$ and $1 \leq i, j \leq n$. Therefore, there exist maximal submodules $V_{n,i}$ of $U_{n,i}$, $n = 2, 3, ...$ and $1 \leq i \leq n$, such that $U_{n,i}/V_{n,i} \cong U_{n,j}/V_{n,j}$ for all n, i, j. Set $V = \oplus_{n,i} V_{n,i}$, $\bar{K} = K/V$ and $S_n = U_{n,1}/V_{n,1}$. Clearly, we have $\frac{K}{V} \supset \frac{\oplus U_{n,i}}{\oplus V_{n,i}} \cong \frac{U_{2,1}}{V_{2,1}} \times \frac{U_{2,2}}{V_{2,2}} \times \frac{U_{3,1}}{V_{3,1}} \times ....$Thus, $G\dim_{S_n}(\bar{K}) \geq n$ for all $n = 2, 3, ...$, which contradicts Lemma 3. Therefore, Λ is directly finite. □

As a special case of the above result, we have the following lemma

Lemma 5. *Let R be a ring which satisfies the condition (*). Then $Q^r_{\max}(R)$ is directly finite and hence R is directly finite.*

Proof. It follows directly from the above theorem by taking $K = R$ and $L = Q^r_{\max}(R)$. □

Theorem 6. *Let R be a ring which satisfies the condition (*). Let K be a finitely generated R-module with $L = E(K)$ and $\Lambda = End(L)$. Then the factor ring $\Lambda/J(\Lambda)$ is the direct product of a finite number of matrix rings over abelian regular self-injective rings.*

Proof. By ([10]), $\bar{\Lambda} = \Lambda/J(\Lambda)$ is a von-Neumann regular, right self-injective ring. By Theorem 4, the ring Λ is directly finite and hence so is the ring $\bar{\Lambda}$. In view of (Theorem 10.22, [10]), $\bar{\Lambda} = A_1 \times A_2$ where A_1 is of Type I_f while A_2 is of Type II_f. Assume that $A_2 \neq 0$. Then by (Proposition 10.28, [10]) there exist idempotents $e_2', e_3', ... \in A_2$ such that $(A_2)_{A_2} \cong n(e_n' A_2)$. In particular, for $n = 3$, $(A_2)_{A_2} \cong 3(e_3' A_2)$, and so $A_2 = e_1 A_2 \oplus e_2 A_2 \oplus e_3 A_2$ where $e_1, e_2, e_3 \in A_2 \subseteq \bar{\Lambda}$ are nonzero orthogonal idempotents such that their sum is the identity of the ring A_2 . Clearly, $e_i \bar{\Lambda} = e_i A_2$, and $e_j A_2 = e_j \bar{\Lambda}$ for all $1 \leq i, j \leq 3$ and so $e_i \bar{\Lambda} \cong e_j \bar{\Lambda}$ Therefore, there exist orthogonal idempotents $u_1, u_2, u_3 \in \Lambda$ such that $u_i + J(\Lambda) = e_i$ for all $i = 1, 2, 3$ (see Corollary 3.9, [18]). In view of (Proposition 21.21, [15]), $u_i \Lambda \cong u_j \Lambda$ for all $1 \leq i, j \leq 3$. Therefore, there exist nonzero cyclic submodules $U_{2i} \subseteq u_i L \cap K$, $i = 1, 2$, such that $U_{21} \cong U_{22}$. By (Corollary 10.9, [10]), $e_3 A_2 e_3 = End(e_3 A_2)$ is of Type II_f and so as above there exist nonzero orthogonal idempotents $f_1, f_2, f_3, f_4 \in e_3 A_2 e_3$ such that $f_i(e_3 A_2 e_3) \cong f_j(e_3 A_2 e_3)$ for all $1 \leq i, j \leq 4$. As before, we lift these orthogonal idempotents to orthogonal idempotents $v_1, v_2, v_3, v_4 \in u_3 \Lambda u_3$ such that $v_i(u_3 \Lambda u_3) \cong v_j(u_3 \Lambda u_3)$ for all i, j. By (Proposition 21.20, [15]), there exist $a \in v_i(u_3 \Lambda u_3) v_j$ and $b \in v_j(u_3 \Lambda u_3) v_i$ such that $v_i = ab$ and $v_j = ba$. Then the mapping which sends $v_i x$ to $b v_i x$, where $x \in \Lambda$, gives isomorphism of $v_i \Lambda$ onto $v_j \Lambda$. So, $v_i \Lambda \cong v_j \Lambda$ for all i, j. Furthermore, we see that there exist nonzero cyclic submodule $U_{3i} \subseteq v_i L \cap K$, $i = 1, 2, 3$ such that $U_{3i} \cong U_{3j}$ for all $1 \leq i, j \leq 3$. Continuing in this fashion, we construct an independent family $\{U_{ij} \mid i = 2, 3, ...; 1 \leq j \leq i\}$ of nonzero cyclic submodules of K such that $U_{ij} \cong U_{ik}$ for all $1 \leq j, k \leq i$; $i = 2, 3, ...$ Therefore, there exist maximal submodules V_{ij} of $U_{ij}, 1 \leq j \leq i$; $i = 2, 3, ...$ such that $U_{ij}/V_{ij} \cong U_{ik}/V_{ik}$ for all i, j, k. Setting $V = \oplus_{i,j} V_{ij}, \bar{K} = K/V$ and $S_i = U_{i1}/V_{i1}$, we get that $G\dim_{S_i}(\bar{K}) \geq i$ for all $i = 2, 3, ...$which contradicts Lemma 3. Therefore, $A_2 = 0$ and $\bar{\Lambda}$ is of Type I_f. Hence $\bar{\Lambda} = \Pi_{i=1}^{\infty} A_i$ where each A_i is of Type I_i ,that is, each A_i is an $i \times i$ matrix ring over an abelian regular self-injective ring (see Theorem 10.24, [10]). Now, we claim that this product must be a finite product. Suppose not, then for any positive integer n there exists an index $m \geq n$ such that $A_m \neq 0$. Now, for any fixed k, we can easily write matrix units $\{e^k_{ij} : 1 \leq i, j \leq k\}$ which are $k \times k$ matrices. So we have an infinite family of nonzero matrix units $\{\{e^k_{ij} | 1 \leq i, j \leq k\} \mid k = 2, 3, ...\} \subseteq \bar{\Lambda}$. Now, since $K \subseteq_e L$, there exists a nonzero cyclic submodule $U_{k,1}$ of K such that $U_{k,1} \subseteq e^k_{1,1} L \cap K$ and then starting with $U_{k,1}$, we construct an independent family $\{U_{k,i}|k = 2, 3, ..., \ \ 1 \leq i \leq k\}$ of cyclic submodules of K such that $U_{k,i} \cong U_{k,j}$ for all k, i, j (exactly as in the proof of Theorem 4). Therefore, there exist maximal submodules $V_{k,i}$ of $U_{k,i}$, $k = 2, 3, ...$ and $1 \leq i \leq k$, such that $U_{k,i}/V_{k,i} \cong U_{k,j}/V_{k,j}$ for all k, i, j. Setting $V = \oplus_{k,i} V_{k,i}$, $\bar{K} = K/V$ and $S_k = U_{k,1}/V_{k,1}$, we get that $G\dim_{S_k}(\bar{K}) \geq k$ for all $k = 2, 3, ...$which contradicts Lemma 3. Therefore, there exists a positive integer n such that $\bar{\Lambda} = \Pi_{i=1}^{n} A_i$. Thus, $\Lambda/J(\Lambda)$ is the direct product of a finite number of matrix rings over abelian regular self-injective rings. □

As a consequence of the above theorem we have

Lemma 7. *Let R be a right nonsingular ring which satisfies condition (*), then $Q^r_{\max}(R)$ (and therefore R) has a bounded index of nilpotence.*

Proof. In the above theorem by taking $K = R$ and $L = Q^r_{\max}(R)$,we have that $Q^r_{\max}(R)$ is a finite direct product of matrix rings over abelian regular self-injective rings. Therefore, $Q^r_{\max}(R)$ (and hence R) has a bounded index of nilpotence. □

Naturally, it is of interest to determine rings for which the condition (*) implies that the ring is right noetherian. We know that it is true for commutative rings [3]. Under the assumption that the ring R is right $q.f.d.$, it was also shown in [3] that R is right noetherian if and only if R satisfies the condition (*). We close by proving that a von-Neumann regular ring R is noetherian if and only if R satisfies a somewhat stronger condition than the condition(*). We do not know whether the result holds with the condition (*).

Theorem 8. *A von-Neumann regular ring R is noetherian if and only if every essential extension of a direct sum of simple R-modules is quasi-injective.*

Proof. Under our hypothesis it is clear that R satisfies (*). Since R is regular, its injective hull as a right R-module is equal to its maximal right ring of quotients $Q = Q^r_{\max}(R)$. Because $Q \simeq End(Q_R)$, it follows from Theorem 6 that Q is the direct product of a finite number of matrix rings over abelian regular self-injective rings. Therefore, Q and hence R has a bounded index of nilpotence.

Now because R is von-Neumann regular, R is semisimple artinian if and only if Q is semisimple artinian. So, assume R is not semisimple artinian (equivalently not noetherian). Since $Q = \Pi_{i=1}^{k} M_{n_i}(S_i)$, for some $1 \leq j \leq k$, $M_{n_j}(S_j)$, and hence S_j is not semisimple artinian. Thus S_j will have infinitely many orthogonal idempotents which are central because S_j is abelian regular. Therefore, there exists an infinite family $\{A_i^{'}|i = 1, 2, ...\}$ of two-sided ideals of S_j such that $A_i^{'}A_j^{'} = A_i^{'} \cap A_j^{'} = 0$ for all $i \neq j$. This gives an infinite family $\{A_i|i = 1, 2, ...\}$ of nonzero two-sided ideals of R such that $A_iA_j = A_i \cap A_j = 0$ for all $i \neq j$. Now, if A_i is contained in all prime ideals then A_i is nil because the intersection of all prime ideals is the lower nil radical. But, since R is a regular ring, it has no nonzero nil ideals. Thus $A_i = 0$, a contradiction. Therefore, for each index $i \geq 1$ there exists a prime ideal P_i of R such that $A_i \nsubseteq P_i$. Because each prime homomorphic image of a von-Neumann regular ring with bounded index of nilpotence is simple artinian, each R/P_i is a simple artinian ring. Since $(A_i + P_i)/P_i$ is a nonzero ideal of the simple artinian ring R/P_i, $(A_i + P_i)/P_i = R/P_i$. As $A_i/A_i \cap P_i \cong (A_i + P_i)/P_i$, we note that $A_i/A_i \cap P_i$ is a simple artinian ring. Set $P = \oplus_{i=1}^{\infty}(A_i \cap P_i)$. Then $(A_i + P)/P \cong A_i/A_i \cap P = A_i/A_i \cap P_i$ and so $B_i = (A_i + P)/P$ is a simple artinian ring and is an ideal of R/P. Clearly, $\{B_i|i = 1, 2, ...\}$ is an independent family of ideals of R/P. Let D/P be a complement of $\oplus_{i=1}^{\infty} B_i$ in R/P. Then $(\oplus_{i=1}^{\infty} B_i) \oplus D/P \subset_e R/P$. Since R/P is von-Neumann regular, (by Remark 3.6, [12]) we may consider D/P to be a two-sided ideal in

R/P. Factoring out by D/P, we obtain that $(\oplus_{i=1}^{\infty} B_i)$ is essentially embeddable in R/D. Without any loss of generality, we may assume that $(\oplus_{i=1}^{\infty} B_i) \subset_e R/P$. Note that $(R/P)_{R/P}$ also satisfies the property that every essential extension of a direct sum of simple modules is a quasi-injective module. Now, every right ideal I/P of R/P is essential extension of $Soc(R/P) \cap I/P$. Therefore, by assumption I/P is quasi-injective. Hence, every right ideal of R/P is quasi-injective. So R/P is a q-ring [13]. Since R/P is semiprime, $R/P = S \oplus T$, where S is semisimple and T has zero socle (see [13]). But R/P has essential socle. So $T = 0$ and hence R/P is semisimple. Let the composition length of R/P be m. Then $\frac{A_1}{A_1 \cap P_1} \times \frac{A_2}{A_2 \cap P_2} \times ... \times \frac{A_{m+1}}{A_{m+1} \cap P_{m+1}} \cong \frac{A_1 \oplus A_2 \oplus ... \oplus A_{m+1}}{A_1 \cap P_1 \oplus A_2 \cap P_2 \oplus ... \oplus A_{m+1} \cap P_{m+1}} \subset \frac{R}{P}$, a contradiction to the composition length of R/P. Therefore, R must be noetherian. This completes the proof. □

Further study on class of rings satisfying the condition (*) to have some sort of finiteness property remains open.

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