Nonnegative Group-Monotone Matrices and the Minus Partial Order

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Abstract: Bapat, Jain and Snyder previously described a class of nonnegative matrices dominated by a nonnegative idempotent matrix under the minus partial order. In this paper, we improve upon that description by first presenting a more general result that gives the precise structure of nonnegative matrices dominated by a group-monotone matrix under the minus partial order. As a special case we derive the complete class of nonnegative matrices dominated by a nonnegative idempotent matrix that includes the class obtained by Bapat et.al.

1 Introduction

Matrix partial orders have proven to be a fruitful area of research. In this paper, our focus is on the minus partial order studied by several authors. We continue our investigation into the structure of nonnegative matrices A that are dominated by a given nonnegative $\lambda-$monotone matrix B under the minus partial order. The present paper studies the case when B is a nonnegative matrix that possesses a nonnegative group inverse. In [1], Bapat-Jain-Snyder considered the case when the matrix B is an idempotent matrix.

We provide necessary and sufficient conditions for a nonnegative matrix A dominated by a nonnegative matrix B that has a nonnegative group inverse (Theorem 6). In the special case when B is an idempotent matrix, Theorem 8 provides an explicit description of the class of nonnegative matrices A dominated by a nonnegative idempotent matrix B.

2 Definitions and Preliminary Results

In this paper, all matrices have real entries. A matrix $A = [a_{ij}]$ is *nonnegative* if $a_{ij} \geq 0$ for all i, j which is denoted $A \geq 0$. Likewise, $A = [a_{ij}]$ is *positive* if $a_{ij} > 0$ for all i, j which is written $A > 0$. The *transpose* of A is denoted by A^T.

Let A be an $m \times n$ matrix. Consider the following equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$ and (5) $AX = XA$ where X is an $n \times m$ matrix and T denotes the *transpose*. For a matrix A and a non-empty subset λ of $\{1, 2, 3, 4, 5\}$, X is called a λ-*inverse* of A if X satisfies equations (i) for $i \in \lambda$. A *{1}-inverse* of A will be written as A^-. The system of equations (1), (2) and (5) has a unique solution X, called the group inverse of A, if and only if $rank(A) = rank(A^2)$. The group inverse of A is denoted by $A^\#$. A matrix A is *group monotone* if $A^\#$ exists and is nonnegative.

A matrix J is a direct sum of matrices $J_1, ..., J_r$, written $J = J_1 \oplus \cdots \oplus J_r$, if

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_r \end{bmatrix}.$$

If A, B are $m \times n$ matrices then we say that A is dominated by B in the minus partial order, written $A \leq^- B$, if $rank\ B = Rank\ A + rank(B - A)$. Theorem 1 will provide us with several equivalent statements.

We begin by stating some of the known facts and preliminary results that are referred to throughout the paper.

Theorem 1 *([8], Lemma 1.2) Let A and B be $m \times n$ matrices. Then the following conditions are equivalent*

1. $A \leq^- B$
2. *There exists a {1}-inverse A^- of A such that $(B-A)A^- = 0$ and $A^-(B-A) = 0$.*
3. *Every {1}-inverse of B is a {1}-inverse of A.*
4. *Every {1}-inverse B^- of B satisfies $AB^-(B-A) = 0$ and $(B-A)B^-A = 0$. In other words, the parallel sum of A and $B - A$ is zero.*

Theorem 2 *([4], Theorem 2). If E is a nonnegative idempotent matrix of rank r, then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where all the diagonal blocks are square; J is a direct sum of matrices $x_i y_i^T$, $x_i > 0$, $y_i > 0$ and $y_i^T x_i = 1$, $i = 1, 2, \ldots, r$; and C, D are nonnegative matrices of suitable sizes.

Lemma 3 *([1], Lemma 3) Let A, E be $n \times n$ matrices such that $E^2 = E$, and suppose that $A \leq^- E$. Then A is idempotent and $AE = A = EA$.*

The next lemma gives the converse when A is also an idempotent.

Lemma 4 *Let A and B be $n \times n$ idempotent matrices. $A \leq^- B$ if and only if $AB = A = BA$.*

Proof. Let A, B be idempotent where $AB = A = BA$. As A is idempotent, $AAA = A^2A = AA = A^2 = A$ so A is its own $\{1\}$-inverse. Now by Theorem 1, we will prove the equivalent condition that there exists a $\{1\}$-inverse A^- of A such that $(B - A)A^- = 0$ and $A^-(B - A) = 0$ to show that $A \leq^- B$. So choose $A = A^-$. Then $(B - A)A^- = (B - A)A = BA - A^2 = BA - A = 0$ as $BA = A$. Also $A^-(B - A) = A(B - A) = AB - A^2 = AB - A = 0$ as $AB = A$. Thus by Theorem 1, $A \leq^- B$ as required. The converse follows by the previous lemma. ■

Theorem 5 *([6], Theorem 5.2) Let A be a nonnegative matrix and let $A^{(1,2)} = p(A) \geq 0$, where $p(A) = \sum_{i=1}^{k} \alpha_i A^{m_i}$, $\alpha_i \neq 0$, $m_i \geq 0$. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I) βxy^T, where x and y are positive unit vectors with $y^T x = 1$ and β is a positive root of $\sum_{i=1}^{k} \alpha_i t^{m_i+1} = 1$

$$(II) \quad \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

where x_i,y_i are positive unit vectors of the same order with $y_i^T x_i = 1$; x_i and x_j, $i \neq j$ are not necessarily of the same order. $\beta_{12}, \ldots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d \mid m_i + 1$ for some m_i such that the product $\beta_{12}\beta_{23} \cdots \beta_{d1}$ is a common root of the following system of at most d equations in t

$$\sum_{d \in \Lambda_0} \alpha_i t^{\frac{(m_i+1)}{d}} = 1$$

$$\sum_{d\in\Lambda_k} \alpha_i t^{\frac{(m_i+1-k)}{d}} = 0 \quad k \in \{1, 2, \ldots, d-1\}$$

where

$$\Lambda_k = \{d : d \mid m_i + 1 - k,\ d \neq 1\} \quad k \in \{0, 1, \ldots, d-1\}$$

with the understanding that if some $\Lambda_k = \emptyset$ then the corresponding equation is absent.

Conversely, suppose we have, for some permutation matrix P,

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where C, D are arbitrary nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both)

$(I^{'})$ βxy^T, $\beta > 0$ where x,y are positive vectors with $y^T x = 1$.

$$(II^{'}) \quad \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

where $\beta_{ij} \geq 0$, x_i and y_i are positive vectors with $y_i^T x_i = 1$. Then $A^{(1,2)} \geq 0$ and is equal to some polynomial in A with scalar coefficients.

3 Group-Monotone Matrices

Theorem 6 *Let A, B be $n \times n$ nonnegative matrices such that $B^{\#} \geq 0$, rank $B = r$ and rank $A = s$. Then $A \leq^- B$ if and only if there exists a permutation matrix P such that*

$$PBP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PAP^T = \begin{bmatrix} A_{11} & A_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CA_{11} & CA_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of appropriate sizes and J is a direct sum of the following types (not necessarily both):

(I) βxy^T, where x and y are positive unit vectors with $y^T x = 1, \beta > 0$.

$$(II) \quad \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

with $\beta_{ij} > 0$ *where* x_i,y_i *are positive unit vectors of the same order with* $y_i^T x_i = 1$; x_i *and* x_j, $i \neq j$ *are not necessarily of the same order and*

$$A_{11} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{bmatrix} \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}$$

where $\alpha_{ij} \geq 0$ *and* $rank\ A_{11} + rank\ (J - A_{11}) = rank\ J$.

Proof. Let A, B be $n \times n$ nonnegative matrices such that $rank\ B = r$ and $rank\ A = s$, $s \leq r$. As $B^{\#} \geq 0$, by Theorem 5 [1] there exists a permutation matrix P such that

$$PBP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is of type (I) or (II). For any $\{1\} - inverse$ B^- of B , $PBP^T = PBB^-BP^T = PBP^TPB^-P^TPBP^T$. Clearly we can choose

$$PB^-P^T = \begin{bmatrix} J^- & J^-D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^- & CJ^-D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where J^- is a $\{1\} - inverse$ of J.

As $A \leq^- B$, by Theorem 1 [8], $AB^-B = AB^-A = BB^-A = A$. Partitioning PAP^T in conformity with PBP^T yields

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

Now $PBB^-AP^T = PBP^TPB^-P^TPAP^T = PAP^T$ gives

$$\begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J^- & J^-D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^- & CJ^-D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} =$$

$$\begin{bmatrix} JJ^-A_{11} + JJ^-DA_{21} & JJ^-A_{12} + JJ^-DA_{22} & JJ^-A_{13} + JJ^-DA_{23} & JJ^-A_{14} + JJ^-DA_{24} \\ 0 & 0 & 0 & 0 \\ CJJ^-A_{11} + CJJ^-DA_{21} & CJJ^-A_{12} + CJJ^-DA_{22} & CJJ^-A_{13} + CJJ^-DA_{23} & CJJ^-A_{14} + CJJ^-DA_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

Then it follows that $A_{2j} = 0$ and $A_{4j} = 0$ for $j = 1, 2, 3, 4$.

Similarly, $PAB^-BP^T = PAP^TPB^-P^TPBP^T = PAP^T$ gives, $A_{i3} = 0$ and $A_{i4} = 0$ for $i = 1, 2, 3, 4$. Thus

$$PAB^-BP^T = \begin{bmatrix} A_{11}J^-J & A_{11}J^-JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{31}J^-J & A_{31}J^-JD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PBB^-AP^T = \begin{bmatrix} JJ^-A_{11} & JJ^-A_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJJ^-A_{11} & CJJ^-A_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $PAB^-BP^T = PAP^T = PBB^-AP^T$.

We will now use the following equations to determine PAP^T:

$$\begin{aligned} A_{11} &= A_{11}J^-J = JJ^-A_{11} \\ A_{12} &= A_{11}J^-JD = JJ^-A_{12} \\ A_{31} &= A_{31}J^-J = CJJ^-A_{11} \\ A_{32} &= A_{31}J^-JD = CJJ^-A_{12} \end{aligned}$$

From the relations above, $A_{11} = A_{11}J^-J = JJ^-A_{11}$, $A_{12} = A_{11}D$, $A_{31} = CA_{11}$, and $A_{32} = CJJ^-A_{12} = CA_{11}J^-JD = CA_{11}D$. As a result,

$$PAP^T = \begin{bmatrix} A_{11} & A_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CA_{11} & CA_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We claim: $A_{11} \leq^- J$ or equivalently $rank(J) = Rank(A_{11}) + rank(J - A_{11})$. First note that $rank(J) = rank(B) = rank(PBP^T)$ and $rank(A_{11}) = rank(A) = rank(PAP^T)$. Because $rank(PBP^T) = rank(PAP^T) + rank(PBP^T - PAP^T)$ it follows that $rank(J) = rank(A_{11}) + rank(J - A_{11})$. Thus we obtain $A_{11} \leq^- J$.

Recall that $A_{11} = A_{11}J^-J = JJ^-A_{11} = A_{11}J^-A_{11}$ where J is a direct sum of matrices of type (I) and (II). Now if $J^{'}$ is any type (I) summand of J, then $J^{'} = \beta xy^T$ where β is a positive scalar and x, y are positive unit vectors. Choose $(J^{'})^- = \frac{1}{\beta}xy^T$. Thus

$$(J^{'})^-J = \left(\frac{1}{\beta}xy^T\right)(\beta xy^T) = xy^T = (\beta xy^T)\left(\frac{1}{\beta}xy^T\right) = J(J^{'})^-$$

Now if $J^{''}$ is any type (II) summand of J, then

$$J^{''} = \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

for some positive integer d . Now

$$\left(J^{''}\right)^{-} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \frac{1}{\beta_{d1}}x_1y_d^T \\ \frac{1}{\beta_{12}}x_2y_1^T & 0 & & & 0 \\ 0 & \frac{1}{\beta_{23}}x_3y_2^T & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\beta_{d-1,d}}x_dy_{d-1}^T & 0 \end{bmatrix}.$$

Then

$$\left(J^{''}\right)\left(J^{''}\right)^{-} = \begin{bmatrix} x_1y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_dy_d^T \end{bmatrix} = \left(J^{''}\right)^{-}\left(J^{''}\right).$$

Thus

$$J^{-}J = JJ^{-} = \begin{bmatrix} x_1y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_ry_r^T \end{bmatrix}$$

is a block diagonal matrix such that each diagonal block is of rank one where the summands of J are of type (I), (II), or both. Now partition A_{11} in conformity with $J^{-}J = JJ^{-}$. So

$$A_{11} = \begin{bmatrix} A_{11}^{'} & \cdots & \cdots & A_{1r}^{'} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{r1}^{'} & \cdots & \cdots & A_{rr}^{'} \end{bmatrix}.$$

From $A_{11}J^{-}J = A_{11} = JJ^{-}A_{11}$, it follows that $x_iy_i^TA_{ij}^{'} = A_{ij}^{'} = A_{ij}^{'}x_jy_j^T$. Clearly, each $A_{ij}^{'}$ must be of rank 0 or 1. Now $A_{ij}^{'} = x_iy_i^TA_{ij}^{'}x_jy_j^T$. Thus, we may write $A_{ij}^{'} = \alpha_{ij}x_iy_j^T$. Hence

$$A_{11} = \begin{bmatrix} \alpha_{11}x_1y_1^T & \alpha_{12}x_1y_2^T & \cdots & \alpha_{1r}x_1y_r^T \\ \alpha_{21}x_2y_1^T & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{r1}x_ry_1^T & \cdots & \cdots & \alpha_{rr}x_ry_r^T \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{r1} & \cdots & \cdots & \alpha_{rr} \end{bmatrix} \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}.$$

The converse is clear. ■

The following example demonstrates that when $A \leq^- B$, with $B^\#$ nonnegative, $A^\#$ need not be nonnegative.

Example 7 *Let* $B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *where* $rank(B) = 3$ *and* $rank(A) = 2$. *Obviously both* $A, B \geq 0$. *Now* $B^\# = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$ *We proceed to show that* A *is of the form stated in the theorem and* $A \leq^- B$. *There exists a permutation matrix* $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *such that*

$$PBP^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PAP^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ where } C, D = 0,$$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ is a type } (II) \text{ matrix}$$

$$\text{and } A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In addition, $(J - A_{11}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$

Therefore, rank A_{11} + *rank* $(J - A_{11}) = 2+1 = 3 =$ *rank* J. *Thus* $A \leq^- B$. *Furthermore, rank* $(A^2) = 2 =$ *rank* (A) *and so* $A^\#$ *exists.*

$$\text{But } A^\# = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \not\geq 0.$$

4 Idempotent Matrices

As a special case of the previous theorem, we provide necessary and sufficient conditions that give the structure of a nonnegative matrix A satisfying $A \leq^- B$ where B is a nonnegative idempotent. The condition obtained in this situation is simpler to verify.

Theorem 8 *Let* A, B *be* $n \times n$ *nonnegative matrices such that* $B^2 = B$, *rank* $B = r$ *and rank* $A = s$. *Then* $A \leq^- B$ *if and only if there exists a permutation matrix* P *such that*

$$PBP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PAP^T = \begin{bmatrix} A_{11} & A_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CA_{11} & CA_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D *are nonnegative matrices of appropriate sizes and* J *is a direct sum of matrices* $x_i y_i^T$ *where* x_i, y_i *are positive unit vectors of the same order with* $y_i^T x_i = 1$; x_i *and* x_j, $i \neq j$ *are not necessarily of the same order and*

$$A_{11} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} E \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}$$

where E *is a nonnegative idempotent* $r \times r$ *matrix.*

Proof. As $B \geq 0$ and B is idempotent, B is its own group inverse. By assumption $A \leq^- B$ and so by Theorem 6

$$PBP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PAP^T = \begin{bmatrix} A_{11} & A_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CA_{11} & CA_{11}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of appropriate sizes and J is a direct sum of matrices $x_i y_i^T$ where x_i, y_i are positive unit vectors of the same order with

$y_i^T x_i = 1$; x_i and x_j, $i \neq j$ are not necessarily of the same order and

$$A_{11} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{bmatrix} \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}.$$

As B is idempotent, it follows from Lemma 3 that A is idempotent and hence PAP^T is idempotent. Thus $A_{11}^2 = A_{11}$ because $y_i^T x_i = 1$. Furthermore,since $\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix}$ has a left inverse and $\begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}$ has a right inverse,

$$E = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{bmatrix}$$

is also an idempotent.

To prove the converse, we first show that $A_{11} \leq^- J$, i.e. $rank(J) = rank(A_{11}) + rank(J - A_{11})$. We have by assumption that $rank\ B = r$ and $rank\ A = s$. It follows that $rank\ PBP^T = r$ and $rank\ PAP^T = s$. Now as the $rank\ PBP^T$ is completely determined by J and $rank\ PAP^T$ is completely determined by A_{11}, $rank\ J = r$ and $rank\ A_{11} = s$. Now $J - A_{11} =$

$$\begin{bmatrix} x_1 y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r y_r^T \end{bmatrix} - \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} E \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix} - \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} E \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} [I - E] \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}.$$

Thus $rank\left(\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} [I-E] \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}\right) \leq rank\,([I-E])$.

But as

$$\begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix} \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} [I-E] \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix} \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} = [I-E]\,,$$

$$rank\left(\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} [I-E] \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^T \end{bmatrix}\right) = rank\,([I-E])\,.$$

Since an idempotent matrix is diagonalizable, there exists an invertible matrix U such that

$$U^{-1}EU = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ 0 & & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

where there are exactly s 1's ones along the diagonal as $rank\ E = s$. Now $U^{-1}[I-E]\,U = U^{-1}IU - U^{-1}EU = I - U^{-1}EU = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} -$ $\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ 0 & & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 0 & 0 & \vdots \\ 0 & & 0 & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$. Hence the $rank(J-A_{11}) = r-s$. Thus $rank(J) = r = s + (r-s) = rank(A_{11}) + rank(J-A_{11})$. This yields $A_{11} \leq^- J$ and hence $A \leq^- B$ as required. ■

We present the following example to demonstrate the structure described in the previous theorem. In this case, B is a doubly stochastic idempotent matrix.

Example 9 *Let* $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *and* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. *Clearly* $A \leq^- B$. *Following the notation in the theorem* $P = I$. *We may express*

$$B = \begin{bmatrix} x_1 y_1^T & 0 & 0 \\ 0 & x_2 y_2^T & 0 \\ 0 & 0 & x_3 y_3^T \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} y_1^T & 0 & 0 \\ 0 & y_2^T & 0 \\ 0 & 0 & y_3^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and clearly $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *where* $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is an idempotent.*

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