

A NOTE ON ALMOST INJECTIVE MODULES

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ABSTRACT. We give some new properties of almost injective modules and their endomorphism rings, and also provide conditions as to when a direct sum of almost injective (or CS) modules is again almost injective (or CS) in some special cases..

1. INTRODUCTION

Let M and N be two right R -modules. As defined by Baba [1] M is called almost N -injective if for each submodule X of N and each homomorphism $f : X \rightarrow M$, either there exists g such that diagram (1) commutes or there exists h such that diagram (2) commutes, where

$$(1) \quad \begin{array}{ccc} 0 & \longrightarrow & X \xrightarrow{i} N \\ & & \downarrow f \quad \swarrow g \\ & & M \end{array}, \quad (2) \quad \begin{array}{ccc} 0 & \longrightarrow & X \xrightarrow{i} N = N_1 \oplus N_2 \\ & & \downarrow \quad \quad \downarrow \pi \\ & & M \xrightarrow{h} N_1 \end{array},$$

N_1 is a nonzero direct summand of N , and $\pi : N \rightarrow N_1$ is a projection onto N_1 . Henceforth, these diagrams will be referred to as diagram (1) and diagram (2), respectively. M is called almost self-injective if M is almost M -injective. A ring R is called right almost self-injective if it is almost self-injective as a right module over itself. Left almost self-injective rings are defined, similarly. Although, this concept has been studied for more than a decade, we find that a number of interesting and useful properties have remained unnoticed. Theorem 5 shows that the endomorphism ring of an indecomposable almost self-injective module is local. Moreover, the endomorphism ring of a uniserial almost self-injective right module is left uniserial (Corollary 7). Also, it is shown that a finite direct sum of indecomposable almost self-injective modules is almost self-injective if the indecomposable modules are relatively almost injective (Remark 17).

Throughout this note, unless otherwise stated, R will denote a ring with identity $1 \neq 0$ and all modules are unital right modules. A module M is called CS if each complement submodule is a direct summand of M . If M^n is CS for every n , then M is called finitely \sum -CS. A module M is called quasi-continuous (also known as π -injective) if for any two submodules M_1

Mathematics Subject Classification. 16D10, 16D50.

Key words and phrases. Almost Injective Modules, Local Rings, Uniserial Rings, Direct sum of Almost Injective (or CS) Modules, Uniform Modules, Eessential Extensions.

and M_2 of M with $M_1 \cap M_2 = 0$, each projection $\pi_i : M_1 \oplus M_2 \rightarrow M_i$, $i = 1, 2$, can be extended to an endomorphism of M . It is well known that a quasi-continuous module is CS. A ring R is said to be a right CS ring if the right R -module R is CS. Left CS rings are defined, similarly. R is called Utumi if its right maximal quotient ring coincides with its left maximal quotient ring.

A decomposition $M = \bigoplus_{i \in I} M_i$ is called exchangeable if for any direct summand N of M , $M = \bigoplus_{i \in I} M'_i \oplus N$ with $M'_i \leq M_i$ (see [8], Definition 4).

The injective hull and the endomorphism ring of a module M will be denoted by $E(M)$ and $End(M)$, respectively. An essential submodule X of a module M will be denoted by $X \subseteq_e M$. We refer to [4, 7] for all undefined notions used in the text.

2. ALMOST SELF-INJECTIVE MODULES

We begin with a simple fact.

Lemma 1. An indecomposable almost self-injective module is quasi-continuous, hence uniform.

Proof. This is obvious. □

For two uniform modules M and N we give below a characterization as to when M is almost N -injective in terms of their injective hulls.

Proposition 2. Let M and N be uniform modules. Then M is almost N -injective if and only if for every $f \in Hom(E(N), E(M))$ either $f(N) \subseteq M$ or f is an isomorphism and $f^{-1}(M) \subseteq N$.

Proof. Assume M is almost N -injective. Let $f \in Hom(E(N), E(M))$ and $X = \{n \in N \mid f(n) \in M\}$. Then $f|_X : X \rightarrow M$. Since M is almost N -injective, either the diagram (1) or the diagram (2) holds. If (1) holds, then there exists $g : N \rightarrow M$ such that $g|_X = f|_X$. We claim $M \cap (g - f)(N) = 0$. Let $m \in M$ such that $m = (g - f)(n)$, for some $n \in N$. Then $f(n) = g(n) - m \in M$. Hence $n \in X$. So $m = g(n) - f(n) = 0$. But $M \subseteq_e E(M)$. Hence $(g - f)(N) = 0$. That is $f(N) \subseteq M$. If (2) holds, then there exists $h : M \rightarrow N$ such that $h \circ f = 1_X$. Hence f is one to one. So f is an isomorphism since $E(N)$ is injective and $E(M)$ is an indecomposable module. Clearly, $h|_{f(X)} = f^{-1}|_{f(X)}$. We claim $N \cap (f^{-1} - h)(M) = 0$. Let $n' \in N$ such that $n' = (f^{-1} - h)(m')$ for some $m' \in M$. Then $f^{-1}(m') = h(m') + n' \in N$. Apply f to both sides, we get $m' = f f^{-1}(m') = f(h(m') + n')$ which implies $m' \in f(X)$. So $n' = (f^{-1} - h)(m') = 0$ because $h|_{f(X)} = f^{-1}|_{f(X)}$ and $m' \in f(X)$. Hence our claim is true. Since $N \subseteq_e E(N)$, $(f^{-1} - h)(M) = 0$. That means $f^{-1}(M) = h(M) \subseteq N$. The converse is clear. □

Proposition 3. Let R be a ring with no nontrivial idempotent. Then R is right almost self-injective if and only if for every $c \in E(R_R)$, either $c \in R$ or there exists $r \in R$ such that $cr = 1$.

Proof. Assume first R is right almost self-injective. Then R_R is uniform by Lemma 1. Let $c \in E(R_R)$ and $l_c : R \rightarrow E(R_R)$ be the left multiplication homomorphism. Then there exists $f : E(R_R) \rightarrow E(R_R)$ such that $l_c|_R = f|_R$. By Proposition 2 either $f(R) \subseteq R$ or f is an isomorphism and $f^{-1}(R) \subseteq R$. If $f(R) \subseteq R$, then $c \in R$. If f is an isomorphism and $f^{-1}(R) \subseteq R$, then there exists $r \in R$ such that $f(r) = 1$. So, $cr = l_c(r) = f(r) = 1$.

Conversely, suppose for every $c \in E(R_R)$, either $c \in R$ or there exists $r \in R$ such that $cr = 1$. We claim that $E(R_R)$ is uniform. For if $e \in \text{End}(E(R_R))$ is an idempotent, then either $e(1) \in R$ or there exists $r \in R$ such that $e(1)r = 1$. If $e(1) \in R$, then $e(1)$ is an idempotent in R and by assumption $e(1) = 0$ or $e(1) = 1$. Hence $e = 0$ or $e = 1_{E(R_R)}$ because $R \subseteq_e E(R_R)$. If $e(1)r = 1$ for some $r \in R$, then $e(r) = 1$. So $e(1) = e(e(r)) = e^2(r) = e(r) = 1$. So $e|_{R_R} = 1_{R_R}$. We proceed to show that $e = 1_{E(R_R)}$. Else, there exists $x \in E(R_R)$ such that $e(x) \neq x$, then $ex - x \neq 0$. Since $R \subseteq_e E(R_R)$, there exists $r' \in R$ such that $(ex - x)r' \neq 0$ and $(ex - x)r' \in R$. Because $(ex - x)r' \in R$, $(ex - x)r' = e(ex - x)r' = 0$, a contradiction to the fact that $(ex - x)r' \neq 0$. Therefore, $e = 1_{E(R_R)}$. This proves $E(R_R)$ is indecomposable and hence uniform. Thus, R_R is uniform. Now let $f \in \text{End}(E(R_R))$. Then by assumption either $f(1) \in R$ or $f(r) = 1$ for some $r \in R$. $f(1) \in R$ implies $f(R) \subseteq R$. If $f(r) = 1$ for some $r \in R$, then $f|_{rR} : rR \rightarrow R$ is an isomorphism. Because $E(R_R)$ is uniform and injective, f is an isomorphism on $E(R_R)$ and $f^{-1}(R) = rR \subseteq R$. By Proposition 2 R is almost self-injective. \square

Corollary 4. Let D be a domain and Q its maximal right ring of quotient. Then D is right almost self-injective if and only if for every $c \in Q$, either c or $c^{-1} \in D$.

It is known that the endomorphism ring of an indecomposable quasi-injective (more generally continuous) module is local. We prove an analogous result for indecomposable almost self-injective module.

Theorem 5. If M is an indecomposable almost self-injective module, then $\text{End}(M)$ is local.

For a proof of this theorem, we first prove the following two lemmas.

Lemma 6. Let M be an indecomposable almost self-injective module. Then for every $f, g \in S = \text{End}(M)$, (i) if $\ker(f) \subsetneq \ker(g)$, then $Sg \subsetneq Sf$, (ii) if $\ker(f) = \ker(g)$, then $Sf \subseteq Sg$ or $Sg \subseteq Sf$.

Proof. Define $\varphi : f(M) \longrightarrow g(M)$ by $\varphi(f(m)) = g(m)$. Clearly, φ is a well defined R -homomorphism. (i) We have $\ker(f) \subsetneq \ker(g)$. Then φ is not a one to one map. By assumption φ can be extended to M . Hence there exists $\psi \in S$ such that $\psi(f(m)) = \varphi(f(m))$ for every $m \in M$. Thus $g(m) = (\psi \circ f)(m)$ for every $m \in M$. Consequently $Sg \subsetneq Sf$. (ii) Let $\ker(f) = \ker(g)$. In this case φ is one to one. So either φ can be extended to an endomorphism $\psi \in S$ or there exists $\eta \in S$ such that $\eta \circ \varphi = 1_{f(M)}$. If $\varphi = \psi$ on $f(M)$, then as above $Sg \subseteq Sf$. If $\eta \circ \varphi = 1_{f(M)}$, then $f(m) = (\eta \circ \varphi)(f(m)) = \eta(\varphi(f(m))) = \eta(g(m)) = (\eta \circ g)(m)$ for every $m \in M$. Thus $Sf \subseteq Sg$. \square

Corollary 7. Let M be a uniserial almost self-injective right R -module. Then $\text{End}(M)$ is left uniserial.

Lemma 8. Let M be an indecomposable almost self-injective module and let $S = \text{End}(M)$. Then the left ideal H of S generated by non-isomorphic monomorphisms in S is a two-sided ideal.

Proof. It is enough to show that $fg \in H$ for each $g \in S$ and for each non-isomorphism $f \in S$ with $\ker(f) = 0$. If $\ker(fg) \neq 0$, then by Lemma 6 $fg \in H$. Now assume that fg is 1-1. If fg were an isomorphism f would be onto, a contradiction. Thus $fg \in H$. \square

Proof of Theorem 5. Let $S = \text{End}(M)$. Then S has no idempotents other than 1 and 0. Recall that since M is indecomposable almost self-injective, it is uniform (Lemma 1). Let F be the set of all non-isomorphic monomorphisms in S . If F is empty, then $\varphi \in S$ is an isomorphism if and only if $\text{Ker}(\varphi) = 0$. Let $h + g \in U(S)$, where $U(S)$ is the group of units of S . Since M is uniform, either $\ker(h) = 0$ or $\ker(g) = 0$. This means either h or g is an isomorphism. Hence S is local. Suppose F is not empty. Let $H = \sum_{f \in F} Sf$. By Lemma 6, $S \setminus U(S) \subset H$. Now let $h \in H$. We show that h

is not invertible. Write $h = \sum_{i=1}^n g_i f_i$, where $f_i \in F$ and $g_i \in S$. By Lemma 6, Sf_1, Sf_2, \dots, Sf_n are linearly ordered. So, $Sf_1 \subseteq Sf_2 \subseteq \dots \subseteq Sf_n$, after reordering if necessary. Hence $h = gf_n$ for some $g \in S$. Now if h is invertible, then f_n is left invertible. Since S has no nontrivial idempotents, f_n is invertible, a contradiction because $f_n \in F$. Thus $H = S \setminus U(S)$. Since H is a two-sided ideal of S (Lemma 8), it follows that S is local. This completes the proof of the theorem.

Let $M = \bigoplus_{\alpha \in I} M_\alpha$. A submodule N of M is said to be finitely contained (written as *f.c.*) in the direct sum, with respect to the decomposition

$M = \bigoplus_{\alpha \in I} M_\alpha$, if there exists $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq I$ such that $N \subseteq \bigoplus_{i=1}^n M_{\alpha_i}$.

A module M is said to have the CS property of uniform (resp. *f.c.* uniform) module if every uniform (resp. *f.c.* uniform) submodule is essential in a direct summand of M . It is well known that if M has a finite uniform dimension and the CS property of uniform module then M is CS (see [4], Corollary 7.8).

Theorem 9. ([6], *Theorem 10*) Assume $\{M_\alpha\}_{\alpha \in I}$ is a set of completely indecomposable R -modules, each M_α is uniform, and $M = \bigoplus_{\alpha \in I} M_\alpha$. Then the following conditions are equivalent:

- (i) M has the CS property of *f.c.* uniform module.
- (ii) For any pair α, β in I , any homomorphism f of a submodule A_α in M_α to M_β is extended to an element in $\text{Hom}_R(M_\alpha, M_\beta)$ or f^{-1} is extended to an element in $\text{Hom}_R(M_\beta, M_\alpha)$, provided $\ker(f) = 0$.

Since the direct sum of CS module is not necessarily CS, it has been a subject of active research to find conditions as to when the direct sum of an indecomposable family of CS modules is CS. The following remark gives one such condition in terms of almost injectivity.

Remark 10. For a module M which can be expressed as a finite direct sum of indecomposable modules $\{M_i\}_{i=1}^n$ it follows from *Theorem 5* and *Theorem 9* that the following are equivalent: (i) $M \oplus M$ is CS and $\text{End}(M_i)$ is local for each i ; (ii) M is finitely \sum -CS and $\text{End}(M_i)$ is local for each i ; (iii) M_i is almost M_j -injective for every i and j .

By Remark 10, an indecomposable module U is almost self-injective if and only if $U \oplus U$ is CS and $\text{End}(U)$ is local. It is shown in the following example that a CS module with local endomorphism ring need not be almost self-injective and hence need not be finitely \sum -CS.

Example 11. Let $F = \mathbb{Q}(x_1, x_2, \dots, x_n \dots)$, $S = \mathbb{Q}(x_1^2, x_2^2, \dots, x_n^2 \dots)$, and $A = \begin{pmatrix} F & 0 \\ F & S \end{pmatrix}$. Let f be the ring homomorphism $f(a) = a$ for all $a \in \mathbb{Q}$ and $f(x_i) = x_i^2$. Let $R = \left\{ \begin{pmatrix} k & 0 \\ k' & f(k) \end{pmatrix} \mid k, k' \in F \right\}$. Then R is a subring of A . The only nontrivial right ideal of R is $\begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$. Thus R is a local right uniserial (hence right CS) ring. If R is right almost self-injective then by Corollary 7 R is left uniserial which is not true. Therefore, R is not right almost self-injective.

Theorem 12. Let M be a nonsingular indecomposable module, $S = \text{End}(M)$, and $Q = \text{End}(E(M))$. Then the following are equivalent:

- (i) M is almost self-injective;
- (ii) For every $f \in Q$ either f or $f^{-1} \in S$;
- (iii) S is a left valuation and right ore domain;
- (iv) S is right almost self-injective;
- (v) S is local and $S \oplus S$ is CS as a right S -module;
- (vi) S is local and finitely \sum -CS as a right S -module;
- (vii) S is Utumi, local and right semihereditary;
- (viii) Left side versions of (iii)-(vii).

Proof. Clearly, S is a domain and Q is its maximal right ring of quotient.

(i) \iff (ii) by Proposition 2.

(ii) \iff (iv) Follows from Corollary 4.

(iv) \iff (v) \iff (vi) by Remark 10.

(iv) \implies (iii) Since S is a domain, we have by Lemma 6 either $Sf \subseteq Sg$ or $Sg \subseteq Sf$ for any f and $g \in S$. Therefore, S is left valuation. By Lemma 1 S is right ore.

(iii) \implies (ii) Let $f \in Q$. Then there exists a nonzero element $g \in S$ such that fg is a nonzero element in S . So, either $Sg \subseteq Sfg$ or $Sfg \subseteq Sg$ because S is left valuation. If $f \notin S$, then $Sfg \not\subseteq Sg$ since S is a right ore domain. Therefore, $Sg \subseteq Sfg$. In particular, $g = hfg$ for some $h \in S$. So, $(1 - hf)g = 0$. Hence, $hf = 1$. This means $f^{-1} = h \in S$.

(vi) \iff (vii) Follows from [3], Theorem 4.9.

(viii) \iff (i) Follows by the symmetry of conditions in (ii). \square

By the symmetry of (iii) in Theorem 12, we have the following Corollary.

Corollary 13. Let D be a domain. D is two-sided valuation if and only if it is left valuation and right ore if and only if it is right or left almost self-injective.

In [5], Hanada, Kuratomi, and Oshiro introduced a generalization of relative injectivity. For two modules M and N , they called M to be generalized N -injective (or N -ojective as in [8]), if for any submodule X of N and any homomorphism $f : X \rightarrow M$, there exist decompositions $N = \overline{N} \oplus \overline{\overline{N}}$, $M = \overline{M} \oplus \overline{\overline{M}}$, a homomorphism $\overline{f} : \overline{N} \rightarrow \overline{M}$, and a monomorphism $g : \overline{\overline{M}} \rightarrow \overline{\overline{N}}$ satisfying the following properties (*), (**)

$$(*) \quad X \subset \overline{N} \oplus g(\overline{\overline{M}})$$

(**) For $x \in X$, we express x in $N = \overline{N} \oplus \overline{\overline{N}}$ as $x = \overline{x} + \overline{\overline{x}}$, where $\overline{x} \in \overline{N}$ and $\overline{\overline{x}} \in \overline{\overline{N}}$. Then $f(x) = \overline{f}(\overline{x}) + \overline{\overline{f}}(\overline{\overline{x}})$, where $\overline{\overline{f}} = g^{-1}$.

M is called generalized self-injective if M is generalized M -injective.

Proposition 14. If M is generalized N -injective, then M is almost N -injective.

Proof. Let X be a submodule of N and $f : X \rightarrow M$ be a homomorphism. Then there exists decompositions $N = \overline{N} \oplus \overline{\overline{N}}$, $M = \overline{M} \oplus \overline{\overline{M}}$, a homomorphism $\overline{f} : \overline{N} \rightarrow \overline{M}$, and a monomorphism $g : \overline{\overline{M}} \rightarrow \overline{\overline{N}}$ satisfying the properties (*), (**). If f can not be extended to N , then $N \neq \overline{N}$. This means $\overline{\overline{N}} \neq 0$. Define $h : M \rightarrow \overline{\overline{N}}$ by $h = g \circ \pi_{\overline{\overline{M}}}$, where $\pi_{\overline{\overline{M}}} : M \rightarrow \overline{\overline{M}}$ is the canonical projection of M onto $\overline{\overline{M}}$ with respect to the decomposition $M = \overline{M} \oplus \overline{\overline{M}}$. For every $x \in X$, express x in $N = \overline{N} \oplus \overline{\overline{N}}$ as $x = \overline{x} + \overline{\overline{x}}$, where $\overline{x} \in \overline{N}$ and $\overline{\overline{x}} \in \overline{\overline{N}}$. Then by (**)

$$\begin{aligned} h \circ f(x) &= h(\overline{f}(\overline{x}) + \overline{\overline{f}}(\overline{\overline{x}})), \text{ where } \overline{\overline{f}} = g^{-1} \\ &= g \circ \pi_{\overline{\overline{M}}}(\overline{f}(\overline{x}) + \overline{\overline{f}}(\overline{\overline{x}})) \\ &= g(\overline{\overline{f}}(\overline{\overline{x}})) = \overline{\overline{x}} = \pi_{\overline{\overline{N}}} \circ i_X(x). \end{aligned}$$

Hence M is almost N -injective. □

Remark 15. Clearly, if M and N are indecomposable modules, then M is almost N -injective if and only if M is generalized N -injective.

For two modules M and N , M is said to be essentially N -injective if for every submodule X of N , any homomorphism $f : X \rightarrow M$ with $\ker(f) \subseteq_e X$, f can be extended to a homomorphism $g : N \rightarrow M$ (see [4], p. 16-17).

Observe that if M is almost N -injective, then M is essentially N -injective. For if X is a submodule of N and $f : X \rightarrow M$ a homomorphism with $\ker(f) \subseteq_e X$, then it follows from [1], Lemma B, that f can be extended to a homomorphism $g : N \rightarrow M$ provided $\ker(f) \subseteq_e N$. Let Y be a complement of X in N and define $h : X \oplus Y \rightarrow M$ by $h(x + y) = f(x)$ for every $x \in X$ and $y \in Y$. Then $\ker(h) = \ker(f) \oplus Y \subseteq_e X \oplus Y \subseteq_e N$. So, $\ker(h) \subseteq_e N$ and hence h can be extended to a homomorphism $g : N \rightarrow M$. Clearly, g is an extension of f .

From the above discussion and Proposition 14 we have the following Proposition of K. Hanada et. al [5]:

Proposition 16. ([5], Proposition 1.4 (2)) If M is generalized N -injective, then M is essentially N -injective.

We close this note by a remark that enable us to produce examples of almost self-injective modules.

Remark 17. Let $\{M_i\}_{i=1}^n$ be a finite set of indecomposable almost self-injective modules. If M_i is almost M_j -injective for any pair i and j in $\{1, 2, \dots, n\}$, then $\bigoplus_{i=1}^n M_i$ is almost self-injective.

Proof. By Lemma 1 each M_i is uniform. By assumption and Remark 15 M_i is generalized M_j -injective for every i and j . Let $M = \bigoplus_{i=1}^n M_i$ and $X = M \oplus M$. Then X is CS and the decomposition $X = \bigoplus_{i=1}^n (M_i \oplus M_i)$ is exchangeable (see [5], Corollary 2.10 and [8], Theorem 13). This implies X is CS and the decomposition $X = M \oplus M$ is exchangeable. Hence M is generalized self-injective (c.f. [5], Theorem 2.1 and [8], Theorem 10) and so $M = \bigoplus_{i=1}^n M_i$ is almost self-injective by Proposition 14. \square

As a consequence of the above remark it follows that for a prime p , $Z/pZ \oplus Z/p^2Z$ is almost self-injective but not quasi-injective. Since any two sided valuation domain D is right and left almost self-injective, we obtain that for all positive integers n , D^n is right and left almost self-injective as a D -module. More generally, if M is any indecomposable almost self-injective module, then M^n is also almost self-injective.

Acknowledgment: We would like to thank Dr. Noyan Er for his carefully reading of the manuscript and his suggestions.

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