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Semiprime CS Group Algebra of Polycyclic-By-Finite Group Without Domains as Summands is Hereditary

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ABSTRACT. Behn showed that if $K[G]$ is a prime group algebra with G polycyclic-by-finite, then $K[G]$ is a CS-ring if and only if $K[G]$ is a pp-ring if and only if G is torsion-free or $G \cong D_\infty$ and $char(K) \neq 2$. As a consequence, such a group algebra $K[G]$ is hereditary excepting possibly when $K[G]$ is a domain. In this paper we show that if $K[G]$ is a semiprime group algebra of polycyclic-by-finite group G and if $K[G]$ has no direct summands that are domains, then $K[G]$ is a CS-ring if and only if $K[G]$ is hereditary if and only if $G/\Delta^+(G) \cong D_\infty$ and $char(K) \neq 2$. Precise structure of a semiprime CS group algebra $K[G]$ of polycyclic-by-finite group G, when K is algebraically closed, is also provided.

1. INTRODUCTION

A ring R is called right CS-ring if every closed right ideal of R is a direct summand. Right selfinjective, continuous, quasi-continuous (= π-injective) rings are CS-rings and have been studied by many authors. But not much is known on CS-group rings. It is well known that the group ring $R[G]$ is selfinjective if and only if R is selfinjective and G is finite. But the corresponding result for CS-group algebras does not hold. For instance, consider the infinite dihedral group D_∞ and a field K with $char(K) \neq 2$. Then the group algebra $K[D_\infty]$ is CS ([**3**], Theorem 3.6). On the other hand if G is a finite group, then the group ring $Z[G]$ is not CS. If $G \cong D_\infty$ and $char(K) \neq 2$, then $gl.dim(K[G]) < \infty$ ([**6**], Theorem 10.3.13). So $gl.dim(K[G]) = h(D_\infty) = 1$ ([**6**], Page 450).

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Thus $K[G]$ is hereditary. Since $K[G]$ is a domain when G is torsion-free, it follows that a prime group algebra $K[G]$ of a polycyclic-by-finite group G which is not a domain is hereditary if and only if it is CS. Thus it is natural to ask when a semiprime CS group algebra $K[G]$ of polycyclic-by-finite group G is hereditary. We show that a semiprime group algebra $K[G]$ of polycyclic-by-finite group G that does not contain a direct summand which is a domain is hereditary if and only if it is CS (Theorem 1). In this paper we also give the precise structure of such a group algebra $K[G]$, when K is an algebraically closed (Theorem 2).

2. NOTATION AND PRELIMINARIES

Throughout, unless otherwise specified, K will denote a field and all modules are unitary. A nonzero module N is said to be an essential submodule of M, if, for every nonzero submodule L of M, $L \cap N \neq 0$. A submodule N of M is called closed or a complement in M if N has no proper essential extension in M. A module M is said to be CS or extending if every closed submodule of M is a summand of M, equivalently, if every nonzero submodule of M is essential in a summand of M. A module M is called finitely $\sum$-CS if finite direct sum of copies of M is CS. A ring R is said to be a right CS-ring (resp. finitely $\sum$-CS ring) if it is CS (resp. finitely $\sum$-CS) as a right module over itself. The group algebra $K[G]$ is prime if and only if G has no nontrivial finite normal subgroup ([**6**], Theorem 4.2.10). If $char(K) = 0$, then $K[G]$ is always semiprime. If $char(K) = p > 0$, then $K[G]$ is semiprime if and only if G has no finite normal subgroups H with $p \mid o(H)$. A twisted group algebra $K^t[G]$ is an associative K-algebra which has a basis $\{\overline{g}, g \in G\}$ and in which the multiplication is defined distributively:

$$\overline{g_1}\,\overline{g_2} = \gamma(g_1, g_2)\overline{g_1 g_2}\,, \qquad g_1, g_2 \in G \text{ and } \gamma(g_1, g_2) \in K^o$$

where K^o is the set of all nonzero elements of K. By choosing $\gamma(g, g^{'}) = 1$ for all $g, g^{'} \in G$, we get the ordinary group algebra $K[G]$ (see [**6**], 1.2). D_∞ as usual stand for the infinite dihedral group generated by two elements a and b with a of infinite order, b of order 2 and $ba = a^{-1}b$. A group G is said to be polycyclic-by-finite if G has a finite subnormal series

$$< 1 >= G_o \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each quotient G_i/G_{i-1} is either finite or cyclic. The number of infinite cyclic quotients which appear in the above series is called the Hirsch number of G, denoted by $h(G)$. This number is invariant for the group (see [**6**]). We may note that, $h(D_\infty) = 1$.

3. SEMIPRIME GROUP RINGS OF POLYCYCLIC-BY-FINITE GROUPS

PROPOSITION 1. *Let K be an algebraically closed field. Then $K^t[N]$ and $K[N]$ are diagonally equivalent, hence $K^t[N] \cong K[N]$, for all twisted group algebras of N over K, where $N \leq D_\infty$.*

PROOF. Nonidentity subgroups of D_∞ are isomorphic to Z, $Z/2Z$ or D_∞. Let N be a subgroup of D_∞ and $K^t[N]$ a twisted group algebra of N over K. If $N = \{1\}$ the result is trivial. Suppose $N \cong Z$ or $Z/2Z$. Then $K^t[N] \cong K[N]$ ([**6**], p. 18). Further, let $N \cong D_\infty$. Write $N =< a, b \mid o(a) = \infty, o(b) = 2$ and $ba = a^{-1}b >$. Since $\overline{b}^2 \in K$ and K is closed under square roots, we can change $\overline{b}$ by an element $b^* \in K^t[N]$ such that ${b^*}^2 = 1$. Now, $b^*\overline{a} = \overline{a}^{-1}b^*k$, for some $k \in K$. Let $t \in K$ such that $t^2 = k$. Set $a^* = t^{-1}\overline{a}$. Then $b^*a^* = {a^*}^{-1}b^*$. Hence $K^t[N] \cong K[N]$ □

LEMMA 1. *([**3**] Theorem 3.6) $K[D_\infty]$ is CS-ring if and only if $char(K) \neq 2$.*

LEMMA 2. *([**1**] Theorem 3.6). Let $K[G]$ be prime with G polycyclic-by-finite. Then the following are equivalent:*

(i) *$K[G]$ is a CS-ring*
(ii) *$K[G]$ is a pp-ring*
(iii) *G is torsion-free or $G \cong D_\infty$ and $char(K) \neq 2$*

LEMMA 3. *([**2**], Corollary 12.18). Let R be a semiprime left and right Goldie ring. Then the following statements are equivalent:*

(i) *R is a left finitely $\sum$-CS*
(ii) *R is a right finitely $\sum$-CS*
(iii) *R is a left semihereditary*
(iv) *R is a right semihereditary.*

In Lemma 2 if $K[G]$ is not a domain, then $G \cong D_\infty$ and hence $K[G] \cong K[D_\infty]$ is hereditary. Therefore, a prime group algebra $K[G]$ of polycyclic-by-finite group G which is not a domain is CS if and only if $K[G]$ is hereditary (by Lemma 3). The Theorem that follows extends the above stated result to a semiprime CS group algebra.

THEOREM 1. *Let $K[G]$ be a semiprime group algebra of a polycyclic-by-finite group G. Suppose $K[G]$ has no ring direct summand which is domain. Then the following are equivalent:*

(i) $K[G]$ *is finitely* $\sum$-CS
(ii) $K[G]$ *is* CS
(iii) $G/\Delta^+(G) \cong D_\infty$ *and* $char(K) \neq 2$
(iv) $K[G]$ *is hereditary*

PROOF. (i) $\Longrightarrow$ (ii) is obvious

(ii) $\Longrightarrow$ (iii) Put $H = \Delta^+(G)$. It is known that $H = \cup N$, where $N \lhd G$ and $o(N) < \infty$. So $H \lhd G$ and $o(H) < \infty$ ([**6**], Lemma 4.1.5(iii)). Hence G/H is a polycyclic-by-finite group having no non-trivial finite normal subgroup. Thus $K[G/H]$ is prime. If $char(K) = p > 0$, then $p \nmid o(H)$ since $K[G]$ is semiprime. Hence in either case we have $o(H)$ is invertible in K. Let $e = o(H)^{-1} \sum_{h\in H} h$. Then e is a central idempotent in $K[G]$. Now,

$$1 - e = 1 - o(H)^{-1} \sum_{h\in H} h = o(H)^{-1} \sum_{h\in H}(1 - h) \in \omega\,(H).$$

So $(1 - e)K[G] \subseteq \omega\,(H)$. Conversely, if $h \in H$, then

$$(1 - h) = (e + (1 - e))\,(1 - h) = (1 - e)\,(1 - h) \in (1 - e)\,K[G],$$

which implies $\omega\,(H) \subseteq (1 - e)\,K[G]$. Hence $\omega\,(H) = (1 - e)\ K[G]$.

So, $K[G/H] \cong K[G]/\omega(H) = K[G]/(1 - e)K[G] \cong eK[G]$.

Since e is a central idempotent in $K[G]$ and $K[G]$ is CS-ring, $eK[G]$ is a CS-ring. Hence $K[G/H]$ is a prime CS-group algebra which is not a domain with G/H polycyclic-by-finite. So by Lemma 1 and Lemma 2, $G/H \cong D_\infty$ and $char\,(K) \neq 2$.

(iii) $\Longrightarrow$ (iv) Let H be as above. Then $gl.\dim K[H] = 0$ since $K[H]$ is semisimple artinian. Also $G/H \cong D_\infty$ and $gl.\dim K[D_\infty] < \infty$ since $char\,(K) \neq 2$ ([**6**], Theorem10.3.13). So by ([**6**], Theorem 10.3.9) $gl.\dim K[G] \leq gl.\dim K[G/H] + gl.\dim K[H]$. So $gl.\dim K[G] < \infty$. Hence $gl.\dim K[G] = h(G) = h(D_\infty) + h(H) = 1 + 0 = 1$ ([**6**], Lemma 10.2.10 and p.450). Thus $K[G]$ is hereditary.

(iv) $\Longrightarrow$ (i) follows from Lemma 3 □

The following lemma will be needed in the next Theorem.

LEMMA 4. *([**5**], Corollary 3.4.10) Let G be a finite group and let K be an algebraically closed field such that $char(K) \nmid O(G)$. Then*

$$K[G] \cong \oplus_{i=1}^{r} M_{n_i}(K)$$

and $n_1^2 + n_2^2 + \cdots + n_r^2 = o(G)$.

The following lemma is a key lemma to prove the next theorem.

LEMMA 5. *(*[**6**]*, Theorem 6.1.9).* *Let* G *be a group, and let* $H \lhd G$. *Suppose* $\{e_1, e_2, ..., e_n\}$ *is a finite* G*-orbit of centrally primitive idempotents of* $K[H]$ *with* $e_1 K[G] \cong M_m(K)$. *Then* $e = e_1 + e_2 + ... + e_n$ *is a central idempotent of* $K[G]$ *and*

$$eK[G] \cong M_{mn}(K^t[G_1/H])$$

where $G_1 \supseteq H$ *is the centralizer of* e_1 *in* G *and* $K^t[G_1/H]$ *is some twisted group ring of* G_1/H.

Now we give the precise structure of the semiprime CS group algebra $K[G]$ of a polycyclic-by-finite group G, when K is algebraically closed and $K[G]$ has no ring direct summands that are domains.

THEOREM 2. *Let* $K[G]$ *be a semiprime* CS *group algebra of a polycyclic-by-finite group* G. *Suppose* $K[G]$ *has no ring direct summand which is domain. If* K *is algebraically closed field, then*

$$K[G] \cong K[D_\infty] \oplus M_{n_1}(K[N_1]) \oplus M_{n_2}(K[N_2]) \oplus \cdots \oplus M_{n_s}(K[N_s])$$

where $N_i \cong D_\infty$ *or* $\mathbb{Z}$.

PROOF. Let $H = \Delta^+(G)$ and $e = o(H)^{-1} \sum_{h \in H} h$. Then $eK[G] \cong K[G/H] \cong K[D_\infty]$ as shown in the proof of Theorem 1. Since $K[G]$ is semiprime and H is a finite normal subgroup of G, we conclude that $K[H]$ is semisimple artinian. Also, by Lemma 4, we have $K[H] \cong \oplus_{i=1}^{r} M_{n_i}(K)$. So $(1-e)K[H] \cong \oplus_{i=1}^{l} M_{n_i}(K)$, where $l \leq r$, after reordering if necessary. So there exists a set $X = \{f_1, f_2, ..., f_l\}$ of centrally primitive orthogonal idempotents in $K[H]$ such that $1 - e = f_1 + f_2 + ... + f_l$ and $f_i K[H] \cong M_{n_i}(K)$, for every $1 \leq i \leq l$. Since $H \lhd G$ and $1-e$ is a central idempotent in $K[G]$, G permutes elements of X. Let s be the number of all G-orbits in X and $\{f_{i_1}, f_{i_2}, ..., f_{i_s}\}$ a subset of X containing exactly one element from each orbit and let $e_j = \sum_{x \in Gf_{i_j}} x$ (the sum of all idempotents in the orbit Gf_{i_j}). Then by Lemma 5 each e_j is a centeral idempotent of $K[G]$. Since $1 - e = e_1 + e_2 + ... + e_s$, we have

$$(1-e)K[G] = e_1 K[G] \oplus e_2 K[G] \cdots \oplus e_s K[G]$$

as a ring direct sum. For each j, $e_jK[G] \cong M_{n_j}(K^t[G_j/H])$, where $G_j \supseteq H$ is the centralizer of e_j in G and $K^t[G_j/H]$is some twisted group ring of G_j/H (Lemma 5). Because $G_1/H < G/H \cong D_\infty$, $K^t[G_j/H] \cong K[G_j/H]$ (Proposition 1). Hence

$$(1-e)K[G] \cong M_{n_1}(K[G_1/H]) \oplus M_{n_2}(K[G_2/H]) \oplus \cdots \oplus M_{n_s}(K[G_s/H]).$$

For each j, the index $[G : G_j] = \mid Gf_{i_j} \mid < \infty$ and also $o(H) < \infty$. So G_j/H is infinite. But infinite subgroups of D_∞ are eihter infinite cyclic or isomorphic to D_∞, we obtain

$$(1-e)K[G] \cong M_{n_1}(K[N_1]) \oplus M_{n_2}(K[N_2]) \oplus \cdots \oplus M_{n_s}(K[N_s])$$

where $N_i \cong D_\infty$ or $\mathbb{Z}$.
This proves,

$$K[G] \cong K[D_\infty] \oplus M_{n_1}(K[N_1]) \oplus M_{n_2}(K[N_2]) \oplus \cdots \oplus M_{n_s}(K[N_s]).$$

where $N_i \cong D_\infty$ or $\mathbb{Z}$. □

Remark. If we assume in Theorem 2 that $K[G]$ has no ring direct summand which is matrix ring over a domain then $K[G]$ is isomorphic to direct sum of matrix rings over $K[D_\infty]$.

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References

[1] Antonio Behn, *Polycyclic Group Rings Whose Principal Ideals Are Projective,* J. Algebra 232, 697-707 (2000)
[2] Nguyen Viet Dung, Dinh Van Huynh, Patrick F. Smith and Robert Wisbauer, *Extending Modules,* Pitman, London, 1994.
[3] S. K. Jain, P. Kanwar, S. Malik and J. B. Srivastava, $K[D_\infty]$ *is a CS-Algebra,* Proc. Amer. Math. Soc. 128 no. 2, 397-400 (2000)
[4] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag (2001)
[5] Csar Polcino Milies and Sudarshan K. Sehgal, *An Introduction to Group Rings*, Kluwer Academic Publishers, Boston (2002)
[6] Donald S. Passman, *The Algebraic Structure of Group Rings,* John Wiley, NY, 1977.

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