

# Modules which are invariant under monomorphisms of their injective hulls

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## Abstract

In this paper certain injectivity conditions in terms of extensions of monomorphisms are considered. In particular, it is proved that a ring  $R$  is a quasi-Frobenius ring if and only if every monomorphism from any essential right ideal of  $R_R$  into  $R_R^{(\mathbb{N})}$  can be extended to  $R_R$ . Also, known results on pseudo-injective modules are extended. Dinh raised the question if a pseudo-injective CS module is quasi-injective. The following results are obtained:  $M$  is quasi-injective if and only if  $M$  is pseudo-injective and  $M^2$  is CS. Furthermore, a uniform pseudo-injective is quasi-injective. As a consequence of this it is shown that over a right Noetherian ring  $R$ , quasi-injective modules are precisely pseudo-injective CS modules.

## 1 Introduction

Throughout the paper rings are associative with identity and modules are unitary (right) modules. Let  $M$  and  $N$  be two right  $R$ -modules over a ring  $R$ .  $M$  is called (pseudo-)  $N$ -injective if, for any submodule  $A$  of  $N$ , every homomorphism (resp. monomorphism) in  $Hom_R(A, M)$  can be extended to an element of  $Hom_R(N, M)$ .  $M$  is called quasi-injective (pseudo-injective) if it is (pseudo-)  $M$ -injective.  $M$  and  $N$  are called relatively injective if  $M$  is  $N$ -injective and  $N$  is  $M$ -injective. A submodule  $K$  of  $M$  is said to be a

complement in  $M$  of a submodule  $B$  if  $K$  is a maximal submodule among those which have zero intersection with  $B$ . Complement submodules of  $M$  coincide with the submodules of  $M$  which do not have any proper essential extension in  $M$ . Also, if  $A$  is a complement in  $M$  and  $B$  is a complement in  $A$ , then  $B$  is a complement in  $M$ . A CS module is one in which complement submodules are direct summands.  $M$  is called a continuous module if it is a CS module and submodules of  $M$  isomorphic to direct summands of  $M$  are again direct summands. If  $M$  is continuous and  $A$  and  $B$  are two direct summands of  $M$  with  $A \cap B = 0$ , then  $A \oplus B$  is also a direct summand of  $M$ . For other properties of complements and CS/ continuous modules and the proofs of the above mentioned properties, the reader is referred to [5] and [13].

In this paper a weaker form of pseudo- $N$ -injectivity is considered and it is proved, in particular, that a ring  $R$  is quasi-Frobenius if and only if monomorphisms from essential right ideals of  $R$  into  $R^{(\mathbb{N})}$  can be extended to  $R_R$ . Also it is shown that a module  $M$  is invariant under monomorphisms of its injective hull if and only if every monomorphism from any essential submodule of  $M$  can be extended to  $M$ . This extension property is used to characterize when semi-prime/right nonsingular rings are SI (see [9]).

Pseudo-injectivity has been studied by several authors such as Dinh, Jain, Singh, Teply, Tuganbaev and others (see [3], [12], [11], [17], [18], [19]). It was first introduced by Jain and Singh [12]. Teply [18] constructed examples of pseudo-injective modules which are not quasi-injective. In [3] Dinh raised the question if a pseudo-injective CS module is quasi-injective. He stated in [4] that the answer is affirmative if we assume further that  $M$  is nonsingular. In this paper we prove the following:  $M$  is quasi-injective if and only if  $M$  is pseudo-injective and  $M^2$  is CS. Every uniform pseudo-injective module is quasi-injective. Consequently, over a right Noetherian ring  $R$ , quasi-injective modules are precisely pseudo-injective CS modules.

## 2 Essentially pseudo- $N$ -injectivity

In this section we consider a weaker form of pseudo- $N$ -injectivity.

**Definition 1** *Let  $M$  and  $N$  be two modules.  $M$  is said to be essentially pseudo- $N$ -injective if for any essential submodule  $A$  of  $N$ , any monomorphism  $f : A \rightarrow M$  can be extended to some  $g \in \text{Hom}(N, M)$ .  $M$  is called essentially pseudo-injective if  $M$  is essentially pseudo- $M$ -injective.*

Obviously any pseudo- $N$ -injective module is essentially pseudo- $N$ -injective, but the converse is not true in general.

**Example 1** Let  $p$  be a prime. The  $\mathbb{Z}$ -module  $\mathbb{Z}/p^2\mathbb{Z}$  is not pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective since the obvious isomorphism  $\iota : p\mathbb{Z}/p^3\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}$  can not be extended to any element of  $Hom(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z})$ , but it is essentially pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective.

The following proposition provides a characterization of essentially pseudo- $N$ -injectivity.

**Proposition 1** *Let  $M$  and  $N$  be two modules and  $X = M \oplus N$ . The following conditions are equivalent:*

- (i)  $M$  is essentially pseudo- $N$ -injective;
- (ii) For any complement  $K$  in  $X$  of  $M$  with  $K \cap N = 0$ ,  $M \oplus K = X$ .

*Proof.* (i)  $\Rightarrow$  (ii) Let  $K$  be a complement in  $X$  of  $M$  with  $K \cap N = 0$ , and  $\pi_M : M \oplus N \rightarrow M$  and  $\pi_N : M \oplus N \rightarrow N$  be the obvious projections. Note that  $M \oplus K = M \oplus \pi_N(K)$  so that  $\pi_N(K)$  is essential in  $N$ .

Now define  $\theta : \pi_N(K) \rightarrow \pi_M(K)$  as follows: For  $k \in K$  with  $k = m + n$  ( $m \in M, n \in N$ ),  $\theta(n) = m$ . Then  $\theta$  is a monomorphism by the  $K \cap N = 0$  assumption. Hence  $\theta$  can be extended to some  $g : N \rightarrow M$ , since  $M$  is essentially pseudo- $N$ -injective. Now let  $T = \{n + g(n) : n \in N\}$ . It is easy to see that  $M \oplus T = X$ . Also,  $T$  contains  $K$  essentially by modularity. Since  $K$  is a complement, this implies  $T = K$ . Now the conclusion follows.

(ii)  $\Rightarrow$  (i) Assume (ii). Let  $A$  be an essential submodule of  $N$  and  $f : A \rightarrow M$  be a monomorphism. Let  $H = \{a - f(a) : a \in A\}$ . Obviously,  $H \cap N = 0$ . Also note that  $M \oplus H = M \oplus \pi_N(H) = M \oplus A$ , which is essential in  $X$ . Let  $K$  be a complement in  $X$  of  $M$  containing  $H$ . By the previous argument and modularity  $H$  is essential in  $K$ , so that  $K \cap N = 0$ . By assumption we have  $M \oplus K = X$ . Now let  $\phi : M \oplus K \rightarrow M$  be the obvious projection. Then the restriction  $\phi|_N$  is the desired extension of  $f$ . The proof is now complete.

**Proposition 2** *If  $M$  is essentially pseudo- $N$ -injective, every direct summand of  $M$  is essentially pseudo- $N$ -injective.*

*Proof.* Let  $X = M \oplus N$  and assume  $M = M_0 \oplus A$ . Let  $K$  be a complement in  $M_0 \oplus N$  of  $M_0$  with  $K \cap N = 0$ . Then  $M \oplus K$  is essential in  $X$ . Since  $K$  is a complement submodule, the preceding argument implies that  $K$  is also a complement in  $X$  of  $M$ . Now by Proposition 1  $M \oplus K = X$ . Then  $M_0 \oplus K = M_0 \oplus N$ , which yields the conclusion again by Proposition 1.

The next example shows that essentially pseudo- $N$ -injectivity is not inherited by direct sums.

**Example 2** Let  $F$  be a field and  $R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix}$ . Consider the  $R$ -modules  $N = \begin{pmatrix} F & F \oplus F \\ 0 & 0 \end{pmatrix}$ ,  $S_1 = \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix}$ ,  $S_2 = \begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix}$ . Then  $S_1$  and  $S_2$  are both essentially pseudo- $N$ -injective. But since the identity map of  $S_1 \oplus S_2$  obviously can not be extended to an element of  $\text{Hom}(N, S_1 \oplus S_2)$ ,  $S_1 \oplus S_2$  is not essentially pseudo- $N$ -injective.

**Proposition 3** *Let  $M$  and  $N$  be two modules. Then the following conditions are equivalent:*

- (i)  $M$  is  $N$ -injective ;
- (ii)  $M$  is essentially pseudo- $N/L$ -injective for every submodule  $L$  of  $N$ .

*Proof.* (i)  $\Rightarrow$  (ii) follows from [13, Proposition 1.3].

(ii)  $\Rightarrow$  (i) Assume  $M$  is essentially pseudo- $N/L$ -injective for every submodule  $L$  of  $N$ . Let  $X = M \oplus N$ ,  $A \subseteq X$  with  $A \cap M = 0$  and  $K$  be a complement in  $X$  of  $M$  containing  $A$ . Also let  $T = K \cap N$ . Since  $(M \oplus K)/K$  is essential in  $X/K$ , then  $(M \oplus K)/T$  is essential in  $X/T$ , and  $K/T \cap N/T = 0$ . Thus it is easy to see that  $K/T$  is a complement in  $X/T$  of  $(M \oplus T)/T$ . Now by assumption and Proposition 1 we have  $(M \oplus T)/T \oplus K/T = X/T$ . Hence  $M \oplus K = X$ . Then by [5, Lemma 7.5]  $M$  is  $N$ -injective.

**Corollary 1**  *$M$  is injective if and only if  $M$  is essentially pseudo- $N$ -injective for any cyclic module  $N$ .*

**Corollary 2** *A nonsingular module  $M$  is injective if and only if it is essentially pseudo- $N$ -injective for any nonsingular cyclic module  $N$ .*

The following result generalizes [3, Theorem 2.2] and [11, Theorem 1].

**Theorem 1** *If  $M \oplus N$  is essentially pseudo- $N$ -injective then  $M$  is  $N$ -injective.*

*Proof.* Call  $X = M \oplus N$ . Let  $A$  and  $K$  be as in the proof of Proposition 3. Let  $\pi : M \oplus N \rightarrow N$  be the obvious projection. Then  $M \oplus K = M \oplus \pi(K)$  and thus  $\pi(K)$  essential in  $N$ . Note that  $K \cong \pi(K)$ . Pick any isomorphism  $f : \pi(K) \rightarrow K$ . By assumption  $f$  can be extended to some monomorphism  $g : N \rightarrow X$ . Then  $g(\pi(K)) = K$  is essential in  $g(N)$ . But since  $K$  is a complement in  $X$ , we must have  $K = g(N)$ , whence  $\pi(K) = N$ . Thus  $M \oplus K = X$ . Now the result follows by [5, Lemma 7.5].

**Corollary 3**  *$M$  is quasi-injective if and only if  $M^2$  is essentially pseudo- $M$ -injective.*

Osofsky proved in [15] that a ring  $R$  is semisimple Artinian if and only if every cyclic right (left)  $R$ -module is injective.

**Corollary 4** *A ring  $R$  is semisimple Artinian if and only if every countably generated right  $R$ -module is essentially pseudo-injective.*

*Proof.* Let  $M$  be a cyclic right  $R$ -module. Then  $(M \oplus R)^{(\mathbb{N})} \cong (M \oplus R)^{(\mathbb{N})} \oplus (M \oplus R)^{(\mathbb{N})}$ , which is countably generated, whence essentially pseudo-injective. Thus  $(M \oplus R^{(\mathbb{N})})^2$  is essentially pseudo- $(M \oplus R^{(\mathbb{N})})$ -injective. Then by Theorem 1,  $(M \oplus R^{(\mathbb{N})})$  is quasi-injective, whence  $R_R$ -injective. Therefore  $M$  is injective. Now the conclusion follows by Osofsky's theorem.

**Corollary 5** ([3, Theorem 2.2]) *If  $M \oplus N$  is pseudo-injective then  $M$  and  $N$  are relatively injective.*

In what follows  $E(M)$  stands for the injective hull of  $M$  and we will consider  $M$  as a submodule of  $E(M)$ . Also we will use the notation  $E_N(M)$  for the submodule of  $E(M)$  generated by all the isomorphic copies of  $N$ . Note that  $E_N(M)$  is invariant under monomorphisms of  $\text{End}(E(M))$  and that  $E_{R_R}(M)$  contains all elements of  $M$  with zero right annihilator in  $R$ .

**Proposition 4**  *$M$  is essentially pseudo- $N$ -injective if and only if  $E_N(M) \subseteq M$ .*

*Proof.* Assume  $E_N(M) \subseteq M$  and let  $B$  be an essential submodule of  $N$ , and  $f : B \rightarrow M$  be a monomorphism. There exists some monomorphism  $g : N \rightarrow E(M)$  such that  $g|_B = f$ . By assumption  $g(N) \subseteq M$ . Thus  $g$  is the desired extension of  $f$ , whence  $M$  is essentially pseudo- $N$ -injective.

Conversely assume that  $M$  is essentially pseudo- $N$ -injective. We will use the same argument as in [13, Lemma 1.13]: Let  $h : N \rightarrow E(M)$  be a monomorphism. Let  $A = h^{-1}(M)$ . Then  $A$  is essential in  $N$ . Thus, by assumption, the restriction  $h|_A$  extends to some  $\theta : N \rightarrow M$ . Now assume  $h(n) \neq \theta(n)$  for some  $n \in N$ . Then  $x = h(n) - \theta(n) \neq 0$ . Since  $M$  is essential in  $E(M)$ , there exists some  $r \in R$  such that  $0 \neq xr = h(nr) - \theta(nr) \in M$ . But then  $h(nr) \in M$  so that  $nr \in A$ . This is a contradiction since  $\theta|_A = h|_A$ . Now the conclusion follows.

**Corollary 6**  *$M$  is essentially pseudo-injective if and only if it is invariant under monomorphisms in  $\text{End}(E(M))$ .*

**Corollary 7** *Let  $\{A_i\}$  be a family of submodules of a module  $N$ ,  $B = \Sigma A_i$  and assume  $M$  is essentially pseudo- $A_i$ -injective for each  $i$ . Then  $M$  is essentially pseudo- $B$ -injective.*

*Proof.* Let  $f : B \rightarrow E(M)$  be a monomorphism. Then  $f(B) = \Sigma f(A_i)$ . By assumption and Proposition 4,  $f(B)$  is contained in  $M$ . Now the conclusion follows again by Proposition 4.

The converse of the Corollary 7 does not hold in general.

**Example 3** Let  $p$  be a prime. It is easy to see that the  $\mathbb{Z}$ -module  $\mathbb{Z}/p^2\mathbb{Z}$  is not essentially pseudo- $\mathbb{Z}/p^3\mathbb{Z}$ -injective, but it is trivially essentially pseudo- $(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$ -injective.

**Corollary 8** *Let  $E$  be an injective module and  $A$  be any submodule of  $E$ . Then  $X = \Sigma\{C \mid C \leq E, C \cong A\}$  is essentially pseudo-injective.*

*Proof.* First note that  $E(X)$  is a summand of  $E$ . As in the proof of Corollary 7, for any monomorphism  $f : X \rightarrow E(X)$ ,  $f(X)$  is contained in  $X$ . The conclusion follows by Proposition 4.

Goodearl defined a right SI-ring to be one over which every singular right module is injective ([9]). Such rings are precisely right nonsingular rings over which singular right modules are semi-simple (see [5]).

**Theorem 2** *Let  $R$  be a ring which is either right nonsingular or semi-prime. The following conditions are equivalent:*

- (i)  $R$  is a right SI-ring;
- (ii) Any two cyclic singular right  $R$ -modules are relatively essentially pseudo-injective;
- (iii) For any two cyclic singular right  $R$ -modules  $B$  and  $C$ ,  $E_B(C) \subseteq C$ .

*Proof.* (i)  $\Rightarrow$  (ii) Trivial.

(ii)  $\Leftrightarrow$  (iii) By Proposition 4.

(ii)  $\Rightarrow$  (i) Assume (ii). Then cyclic singular right  $R$ -modules are relatively injective by Proposition 3. So if  $C$  and  $M$  are singular right  $R$ -modules and  $C$  is cyclic, then  $C$  is  $M$ -injective by the above argument and [13, Proposition 1.4]. This implies, by [5, Corollary 7.14], that all singular right  $R$ -modules are semi-simple.

Now, if  $R$  is right nonsingular, the conclusion immediately follows by the preceding remark and the above argument. Else, assume that  $R$  is semi-prime. Since singular modules are semi-simple,  $Z(R_R)^2 = 0$ , whence  $Z(R_R) = 0$ . Now the conclusion follows by the above argument.

### 3 Pseudo-injectivity

**Proposition 5** *Let  $M$  and  $N$  be two modules and  $X = M \oplus N$ . The following conditions are equivalent:*

- (i)  $M$  is pseudo- $N$ -injective;
- (ii) For any submodule  $A$  of  $X$  with  $A \cap M = A \cap N = 0$ , there exists a submodule  $T$  of  $X$  containing  $A$  with  $M \oplus T = X$ .

*Proof.* (i)  $\Rightarrow$  (ii) Assume (i) and let  $A$  satisfy the assumptions of (ii). Also let  $\pi_M$  and  $\pi_N$  be as in the Proposition 1, and define  $\theta : \pi_N(A) \rightarrow \pi_M(A)$  as follows:  $\theta(\pi_N(a)) = \pi_M(a)$ , for  $a \in A$ . Then, by assumption,  $\theta$  extends to some  $g \in \text{Hom}(N, M)$ . Let  $T = \{n + \theta(n) | n \in N\}$ . Then we have  $M \oplus T = X$  and  $A \subseteq T$ , as required.

(ii)  $\Rightarrow$  (i) Assume (ii). Let  $B$  be a submodule of  $N$  and  $f : B \rightarrow M$  be a monomorphism. Call  $A = \{b - f(b) | b \in B\}$ . Then  $A \cap M = A \cap N = 0$ .

Now, by assumption, there exists a submodule  $T$  of  $X$  containing  $A$  with  $M \oplus T = X$ . Let  $\pi : M \oplus T \rightarrow M$  be the obvious projection. Then the restriction  $\pi|_N$  is the desired extension of  $f$ .

Jain and Singh proved in [12, Theorem 3.7] that for a nonsingular module  $M$  with finite uniform dimension, the following conditions are equivalent: (i)  $M$  is pseudo-injective; (ii)  $M$  is invariant under any monomorphism ("isomorphism" in the terminology of [12]) of  $\text{End}(E(M))$  (i.e.  $M$  is essentially pseudo-injective by Corollary 6). The following result extends it to any module with finite uniform dimension.

**Theorem 3** *A module  $M$  with finite uniform dimension is pseudo-injective if and only if it is essentially pseudo-injective.*

*Proof.* Let  $M$  be essentially pseudo-injective and  $A$  be a submodule of  $M$  with a monomorphism  $f : A \rightarrow M$ . Call  $B = f(A)$ . Pick, by Zorn's Lemma, two submodules  $A'$  and  $B'$  of  $M$  such that  $A \oplus A'$  and  $B \oplus B'$  are essential in  $M$ . Now,  $E(M) = E(A) \oplus E(A') = E(B) \oplus E(B')$  and  $E(A) \cong E(B)$ . Then by [13, Theorem 1.29] and since  $M$  has finite uniform dimension, we have  $E(A') \cong E(B')$ . Thus  $A'$  and  $B'$  have isomorphic essential submodules  $U \subseteq A'$  and  $V \subseteq B'$ . Then  $A \oplus U$  and  $B \oplus V$  are essential submodules of  $M$ . Let  $\phi : U \rightarrow V$  be any isomorphism. Then there exists an isomorphism  $\theta : A \oplus U \rightarrow B \oplus V$  such that  $\theta|_A = f$ . By assumption  $\theta$  extends to some  $g \in \text{End}(M)$ . Obviously,  $g|_A = f$ . Therefore the conclusion follows.

Note that, in [1, Theorem 2.1] Alamelu gives a proof of the equivalence in Theorem 3 without the finite dimension assumption. However the proof is incorrect. In summary, the proof states that, for a module  $M$  which is invariant under monomorphisms of its injective hull, and for any monomorphism  $f : N \rightarrow M$  where  $N$  is a submodule of  $M$ ,  $f$  can be extended to a monomorphism  $f'' : E(M) \rightarrow E(M)$ . This is not correct as the following example shows: Let  $M$  be any directly infinite injective module with  $M = N \oplus B$  where  $M \cong N$  and  $B$  is nonzero. Also let  $f : N \rightarrow M$  be any isomorphism. Obviously  $f$  can not be extended to a monomorphism in  $\text{End}(E(M))$ .

In [6] and [7] Er studied the modules in which isomorphic copies of complements are again complements. These are called SICC-modules in [7]. The following result was proved in [12] for nonsingular modules, but the proof works for an arbitrary pseudo-injective module as well.



**Lemma 1** (*Jain and Singh [12, Lemma 3.1]*) *If  $M$  is pseudo-injective then submodules of  $M$  isomorphic to complements in  $M$  are again complements.*

*Proof.* Let  $K$  be a complement in  $M$  and  $A$  be a submodule of  $M$  with an isomorphism  $f : A \rightarrow K$ . Then  $f$  extends to some  $g \in \text{End}(M)$  by assumption. Pick, by Zorn's Lemma, a complement  $A'$  in  $M$  essentially containing  $A$ . Then the restriction  $g|_{A'}$  is obviously a monomorphism. Hence  $K = g(A)$  is essential in  $g(A')$ . Since  $K$  is a complement this implies  $K = g(A')$ , whence  $A = A'$ . The conclusion follows.

*Remark* Modules in which submodules isomorphic to complements are complements always decompose into relatively injective summands by [7, Lemma 4]. So Corollary 5 also follows from that result and Lemma 1. It is proved in [3, Corollary 2.8] that a pseudo-injective CS module is continuous. This result also follows from Lemma 1 and the definition of CS.

Dinh [3] raised the question whether a CS module  $M$  which is pseudo-injective is quasi-injective, and stated in [4] that the answer is affirmative when  $M$  is furthermore nonsingular. Now we present some partial answers to Dinh's question.

**Theorem 4**  *$M$  is quasi-injective if and only if  $M$  is pseudo-injective and  $M^2$  is CS.*

*Proof.* Assume  $M$  is pseudo-injective and  $M^2$  is CS. Let  $M_1$  and  $M_2$  be two isomorphic copies of  $M$  and  $X = M_1 \oplus M_2$ . Note that  $M$  is continuous by the preceding remark.

First let  $A$  be any complement in  $X$  with  $A \cap M_1 = 0$  and  $A \cap M_2$  essential in  $A$ . There exist submodules  $V$  and  $V'$  of  $M_2$  such that  $V \oplus V' = M_2$  and  $V$  contains  $A \cap M_2$  essentially. Also since  $M^2$  is CS by assumption, we have  $A \oplus A' = X$  for some submodule  $A'$  of  $X$ . Since  $V$  is a direct summand of a continuous module,  $V$  is continuous (see [13]), whence it has exchange property by [13, Theorem 3.4]. Since  $V \cap A$  is essential in  $A$  we have  $V \cap A' = 0$ . Thus we must have  $V \oplus A' = X$ . Hence  $A$  is isomorphic to a summand, namely  $V$  of  $M_2$ .

Now let  $C$  be a submodule of  $X$  such that  $C \cap M_1 = 0$  and pick, by Zorn's Lemma, a complement  $K$  in  $X$  of  $M_1$  containing  $C$ . Again by Zorn's Lemma, choose a complement  $K_1$  in  $K$  of  $K \cap M_2$  and a complement  $K_2$  in

$K$  of  $K_1$  containing  $K \cap M_2$ . Note that  $K \cap M_2$  is essential in  $K_2$  and that  $K_1$  and  $K_2$  are complements in  $X$  by [5, 1.10]. By Proposition 5 there exists some submodule  $T$  of  $X$  containing  $K_1$  with  $M_1 \oplus T = X$ . Then  $T \cong M$  and  $K_1$  is a complement in  $T$ , whence  $K_1$  is isomorphic to a complement in  $M_2$ . Also by the preceding paragraph  $K_2$  is isomorphic to a complement of  $M_2$  too. Now consider the usual projection  $\pi : M_1 \oplus M_2 \rightarrow M_2$ . We have  $M_1 \oplus (K_1 \oplus K_2) = M_1 \oplus (\pi(K_1) \oplus \pi(K_2))$ , where  $\pi(K_i) \cong K_i$ . Hence by continuity of  $M_2$  and the above argument,  $\pi(K_1) \oplus \pi(K_2)$  is a summand of  $M_2$ . Now, since  $K$  is a complement of  $M_1$ ,  $M_1 \oplus K = M_1 \oplus \pi(K)$  is essential in  $X$ . Then  $\pi(K)$  is essential in  $M_2$ . Also, by choice of  $K_i$ ,  $K_1 \oplus K_2$  is essential in  $K$ . Then  $\pi(K_1) \oplus \pi(K_2)$  is essential in  $\pi(K)$ , hence in  $M_2$ . This implies that  $M_2 = \pi(K_1) \oplus \pi(K_2) = \pi(K)$ . Thus  $M_1 \oplus K = X$ . Now it follows by [5, Lemma 7.5] that  $M_1$  is  $M_2$ -injective. The proof is now complete.

The following is a key result.

**Lemma 2** *The following conditions hold:*

- (i) *A uniform pseudo-injective module  $M$  is quasi-injective.*
- (ii) *Let  $M = \bigoplus_{i \in I} M_i$  be a direct sum of uniform modules  $M_i$ .  $M$  is quasi-injective if and only if it is pseudo-injective.*

*Proof.* (i) Let  $A$  be a submodule of  $M$  and  $f : A \rightarrow M$  be a nonzero homomorphism. If  $\text{Ker}(f) = 0$  then  $f$  can be extended to an element of  $\text{End}(M)$  by assumption. So assume  $\text{Ker}(f) \neq 0$ . Let  $\delta = i_A - f$ , where  $i_A : A \rightarrow M$  is the inclusion map. Since  $\text{Ker}(f) \neq 0$  and  $M$  is uniform,  $\text{Ker}(\delta) = 0$ . Then by pseudo-injectivity assumption  $\delta$  can be extended to some  $g \in \text{End}(M)$ . Now  $1 - g$  is obviously an extension of  $f$ . The conclusion follows.

(ii) Let  $M$  be pseudo-injective. Then, by Corollary 5,  $M(I - i)$  is  $M_i$ -injective for all  $i \in I$ . Now by part (i) and since direct summands of pseudo-injectives are obviously pseudo-injective, each  $M_i$  is quasi-injective. Therefore  $M$  is quasi-injective.

**Theorem 5** *Over a right Noetherian ring  $R$ , a module  $M$  is quasi-injective if and only if  $M$  is a pseudo-injective CS-module.*

*Proof.* Let  $M$  be a pseudo-injective CS module. Then  $M$  is a direct sum of uniform submodules by [14]. Now the result follows by Lemma 2.

Before proving the next result, note that  $R$  is called a right countably  $\Sigma$ -CS ring if  $R_R^{(\mathbb{N})}$  is a CS module.

**Theorem 6** *The following conditions are equivalent for a ring  $R$ :*

- (i)  $R$  is a quasi-Frobenius ring;
- (ii) Every projective right  $R$ -module is essentially pseudo- $R_R$ -injective;
- (iii)  $R_R^{(\mathbb{N})}$  is essentially pseudo- $R_R$ -injective;
- (iv)  $R$  is a right countably  $\Sigma$ -CS ring with finite uniform dimension and  $R_R$  is essentially pseudo-injective.

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) are obvious, and (i)  $\Rightarrow$  (iv) follows from the fact that every injective module is CS, and (iii)  $\Rightarrow$  (i) follows by Theorem 1.

(iv)  $\Rightarrow$  (i) Since  $R_R$  has finite uniform dimension, then  $R_R$  is pseudo-injective by Theorem 3. By assumption  $R_R = \bigoplus_{i=1}^n e_i R$  for some uniform right ideals  $e_i R$ . By Corollary 5  $e_i R$  are relatively injective. Also by Lemma 2 each  $e_i R$  is a quasi-injective right  $R$ -module. Thus  $R$  is right self-injective with finite uniform dimension. Hence  $R$  is a semiperfect right countably  $\Sigma$ -CS ring. This implies by [10] that  $R$  is Artinian. Now the conclusion follows.

The following results were proved in [7, Theorem 2, Corollary 4, Theorem 3, Theorem 4] for modules in which submodules isomorphic to complements are complements. Each pseudo-injective module satisfies this property by Lemma 1, whence we have the following corollaries.

**Corollary 9** *Any decomposition of a pseudo-injective module into indecomposable submodules complements summands.*

**Corollary 10** *A essentially pseudo-injective module with finite uniform dimension has the internal cancellation property.*

Recall that every right  $R$ -module over a right Noetherian ring  $R$  is locally Noetherian.

**Corollary 11** *If  $M$  is a locally Noetherian pseudo-injective module, then  $M = A \oplus B$ , where  $A$  is a maximal quasi-injective summand,  $B$  has no quasi-injective summands, and  $A$  and  $B$  have no nonzero isomorphic submodules.*

**Corollary 12** *A locally Noetherian Dedekind-finite pseudo-injective module has internal cancellation property.*

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