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Nonnegative matrices A with $AA^\sharp \geqslant 0$

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Abstract

In this paper we obtain a decomposition of nonnegative matrices A such that $AA^\sharp \geqslant 0$. We then use this characterization to obtain the previous results known for nonnegative matrices A with $A^\sharp \geqslant 0$. We also consider nonnegative matrices A with $A - A^2 \geqslant 0$.

Keywords: Nonnegative matrices; Group inverse; Monotone matrices; Idempotent matrices

1. Introduction

Nonnegative matrices with nonnegative group inverse have been studied by many authors (see for example [3,4,6,8,9]). In another paper [10], we considered nonnegative matrices A with $AA^\dagger$ or $A^\dagger A$ nonnegative. In this paper, we prove analogous results characterizing nonnegative matrices A with $AA^\sharp$ nonnegative. We give an example that shows that this class of nonnegative matrices A with $AA^\sharp$ nonnegative is properly contained in the class of nonnegative A with $A^\sharp$ nonnegative.

A matrix $A = (a_{ij})$ is called nonnegative if $a_{ij} \geqslant 0$ for all i, j and this is expressed as $A \geqslant 0$. A matrix A is called reducible if it is cogredient to $E = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$, where B and D are square matrices, or $A = 0$. Otherwise, A is called irreducible. We denote the spectral radius of a matrix A by $\rho(A)$.

If there exists a matrix X such that $AXA = A$, $XAX = X$, and $AX = XA$, then this is referred to as the group inverse of A. If the group inverse exists, it is unique

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and denoted by $A^{\#}$. It is well known that the group inverse exists if and only if index $A = 1$. We refer to a matrix X such that $AXA = A$ and $AX = XA$ as a $\{1, 5\}$-*inverse* of A and denote it by $A^{(1,5)}$. A matrix is called group-monotone if $A^{\#}$ exists and is nonnegative. In this paper, we consider a weaker condition and only require that A and $AA^{\#}$ be nonnegative.

In Section 2 we prove the main result and obtain as a special case the well-known characterization of nonnegative matrices A with $A^{\#}$ nonnegative. In the last section we consider nonnegative matrices A with $A - A^2 \geqslant 0$.

The reader is referred to [1] for additional definitions and results on generalized inverses.

2. Main result

Theorem 1. *Let A be a nonnegative $n \times n$ matrix of* rank r. *Then the following are equivalent*:

(i) *There exists an $A^{(1,5)}$ such that $AA^{(1,5)} \geqslant 0$.*

(ii) *There exists a permutation matrix P such that*

$$PAP^{\mathrm{T}} = \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal blocks are square, T is a nonnegative $r \times r$ invertible matrix,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix}, \quad Y = \begin{bmatrix} y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & y_2^{\mathrm{T}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^{\mathrm{T}} \end{bmatrix},$$

x_i and y_i are positive unit vectors such that $y_i^{\mathrm{T}} x_i = 1$, and B, C are nonnegative matrices of appropriate size.

(iii) $AA^{\#} \geqslant 0$.

In particular, under any of the above equivalent conditions,

$$PA^{\#}P^{\mathrm{T}} = \begin{bmatrix} XT^{-1}Y & XT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXT^{-1}Y & CXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. (i) $\Rightarrow$ (ii): So, there exists an $A^{(1,5)}$ such that $AA^{(1,5)} \geqslant 0$. Then, $AA^{(1,5)}$ is a nonnegative idempotent. So by Flor [5], there exists a permutation matrix P such that

$$PAA^{(1,5)}P^{\mathrm{T}} = \begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{where } J = \begin{bmatrix} x_1y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_ry_r^{\mathrm{T}} \end{bmatrix},$$

each x_i, y_i are positive vectors with $y_i^{\mathrm{T}}x_i = 1$, matrices $B, C \geqslant 0$ and the zero's in the matrices are zero blocks of appropriate size. Note that rank $AA^{(1,5)} =$ rank $J = r$. Next, we partition PAP^{T} in conformity with $PAA^{(1,5)}P^{\mathrm{T}}$ and let

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

Clearly, $PAP^{\mathrm{T}}PAA^{(1,5)}P^{\mathrm{T}} = PAP^{\mathrm{T}}$ and so,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

This implies

$$\begin{bmatrix} A_{11}J + A_{13}CJ & A_{11}JB + A_{13}CJB & 0 & 0 \\ A_{21}J + A_{23}CJ & A_{21}JB + A_{23}CJB & 0 & 0 \\ A_{31}J + A_{33}CJ & A_{31}JB + A_{33}CJB & 0 & 0 \\ A_{41}J + A_{43}CJ & A_{41}JB + A_{43}CJB & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

By equating corresponding blocks we obtain $A_{i3} = 0$ and $A_{i4} = 0$ for $i = 1, 2, 3, 4$ hence, $A_{i1} = A_{i1}J$, and $A_{i1}JB = A_{i2}$ for $i = 1, 2, 3, 4$. Hence,

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 \end{bmatrix}.$$

Since $PAA^{(1,5)}P^{\mathrm{T}}PAP^{\mathrm{T}} = PAP^{\mathrm{T}}$ we get the following:

$$\begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} JA_{11} + JBA_{21} & JA_{12} + JBA_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJA_{11} + CJBA_{21} & CJA_{12} + CBA_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 \end{bmatrix}.$$

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This yields that $A_{2i} = 0$, $A_{4i} = 0$ for $i = 1, 2$. Hence, $JA_{11} = A_{11}$ and $A_{31} = CJA_{11}$.

Therefore,

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{11}B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CA_{11} & CA_{11}B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $JA_{11} = A_{11} = A_{11}J$. Also, we have that rank $A = $ rank $A_{11} = r$.

We now consider $JA_{11} = A_{11}$. As above, we partition A_{11} in conformity with J and let $A_{11} = (A'_{ij})$. Then

$$\begin{bmatrix} x_1y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_ry_r^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A'_{11} & \cdots & \cdots & A'_{1r} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A'_{r1} & \cdots & \cdots & A'_{rr} \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & \cdots & A'_{1r} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A'_{r1} & \cdots & \cdots & A'_{rr} \end{bmatrix}.$$

So,

$$\begin{bmatrix} x_1y_1^{\mathrm{T}}A'_{11} & \cdots & \cdots & x_1y_1^{\mathrm{T}}A'_{1r} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_ry_r^{\mathrm{T}}A'_{r1} & \cdots & \cdots & x_ry_r^{\mathrm{T}}A'_{rr} \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & \cdots & A'_{1r} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A'_{r1} & \cdots & \cdots & A'_{rr} \end{bmatrix}.$$

Equating corresponding blocks, we have $x_iy_i^{\mathrm{T}}A'_{ij} = A'_{ij}$. Because $A_{11} = (A'_{ij})$ and rank $A_{11} = r$, we must have for each i, at least one j such that A'_{ij} is nonzero. To see this, consider the submatrix $[x_1y_1^{\mathrm{T}}A'_{11} \ \cdots \ \cdots \ x_1y_1^{\mathrm{T}}A'_{1r}]$. Clearly, we have rank $[x_1y_1^{\mathrm{T}}A'_{11} \ \cdots \ \cdots \ x_1y_1^{\mathrm{T}}A'_{1r}] \leqslant 1$. If rank $[x_1y_1^{\mathrm{T}}A'_{11} \ \cdots \ \cdots \ x_1y_1^{\mathrm{T}}A'_{1r}] = 0$, then rank $A_{11} < r$, a contradiction. Thus, rank $[x_1y_1^{\mathrm{T}}A'_{11} \ \cdots \ \cdots \ x_1y_1^{\mathrm{T}}A'_{1r}] = 1$, and this implies that not all of the A'_{1j} are zero. Therefore, as claimed, for each i, there exists at least one j such that $A'_{ij} \neq 0$. For convenience, we choose $A'_{11} \neq 0$. Then, rank $A'_{11} = 1$. Thus, $A'_{11} = u_{11}v_{11}^{\mathrm{T}}$ where u_{11} and v_{11} are vectors such that either $u_{11} \geqslant 0$ and $v_{11} \geqslant 0$, or $u_{11} \leqslant 0$ and $v_{11} \leqslant 0$. We may assume that v_{11} is a unit vector. Then we have $x_1y_1^{\mathrm{T}}u_{11}v_{11}^{\mathrm{T}} = u_{11}v_{11}^{\mathrm{T}}$. So, we multiply by v_{11} on the right and get $x_1y_1^{\mathrm{T}}u_{11} = u_{11}$. But then $\lambda_{11} = y_1^{\mathrm{T}}u_{11}$ is a scalar and so u_{11} is a multiple of x_1. It follows then that if $u_{11} \geqslant 0$, then it is in fact a positive vector. If $u_{11} \leqslant 0$, then $\lambda_{11} < 0$. So let $u_{11} = \lambda_{11}x_1$. Now, we have that $A'_{11} = \lambda_{11}x_1v_{11}^{\mathrm{T}} = x_1v_{11}'^{\mathrm{T}}$ where $v_{11}'^{\mathrm{T}} = \lambda_{11}v_{11}^{\mathrm{T}}$. Thus, we have that $v_{11}'^{\mathrm{T}} \geqslant 0$, because if $v_{11} \leqslant 0$, then $\lambda_{11} < 0$ and if $v_{11} \geqslant 0$, then $\lambda_{11} > 0$. This process can indeed be repeated for each A'_{ij}. So we have

$$A_{11} = \begin{bmatrix} x_1 v_{11}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{1r}'^{\mathrm{T}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_1 v_{r1}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{rr}'^{\mathrm{T}} \end{bmatrix}.$$

We next consider the equation $A_{11} J = A_{11}$. This gives us

$$\begin{bmatrix} x_1 v_{11}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{1r}'^{\mathrm{T}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_1 v_{r1}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{rr}'^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_1 y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r y_r^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} x_1 v_{11}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{1r}'^{\mathrm{T}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_1 v_{r1}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{rr}'^{\mathrm{T}} \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} x_1 v_{11}'^{\mathrm{T}} x_1 y_1^{\mathrm{T}} & \cdots & \cdots & x_1 v_{1r}'^{\mathrm{T}} x_r y_r^{\mathrm{T}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_1 v_{r1}'^{\mathrm{T}} x_1 y_1^{\mathrm{T}} & \cdots & \cdots & x_1 v_{rr}'^{\mathrm{T}} x_r y_r^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} x_1 v_{11}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{1r}'^{\mathrm{T}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ x_1 v_{r1}'^{\mathrm{T}} & \cdots & \cdots & x_1 v_{rr}'^{\mathrm{T}} \end{bmatrix}.$$

So $x_1 v_{11}'^{\mathrm{T}} x_1 y_1^{\mathrm{T}} = x_1 v_{11}'^{\mathrm{T}}$. Multiplying by y_1^{T} on the left gives us $v_{11}'^{\mathrm{T}} x_1 y_1^{\mathrm{T}} = v_{11}'^{\mathrm{T}}$. Since $\alpha_{11} = v_{11}'^{\mathrm{T}} x_1$ is a scalar we have $\alpha_{11} y_1^{\mathrm{T}} = v_{11}'^{\mathrm{T}}$. Finally we now have that

$$A_{11} = \begin{bmatrix} \alpha_{11} x_1 y_1^{\mathrm{T}} & \alpha_{12} x_1 y_2^{\mathrm{T}} & \cdots & \alpha_{1r} x_1 y_r^{\mathrm{T}} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_{r1} x_r y_1^{\mathrm{T}} & \alpha_{r2} x_r y_2^{\mathrm{T}} & \cdots & \alpha_{rr} x_r y_r^{\mathrm{T}} \end{bmatrix},$$

where α_{ij} are nonnegative constants. We then have

$$A_{11} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{bmatrix} \begin{bmatrix} y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^{\mathrm{T}} \end{bmatrix}$$
$$= XTY,$$

say. Thus, because rank $A_{11} = r$, we must have that rank $T \geqslant r$. But, T is an $r \times r$ matrix, and so rank $T = r$. Therefore T is invertible.

So, we have shown

$$PAP^{\mathrm{T}} = \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

proving (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): So, there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal blocks are square, T is a nonnegative invertible matrix,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix}, \quad Y = \begin{bmatrix} y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & y_2^{\mathrm{T}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^{\mathrm{T}} \end{bmatrix},$$

x_i and y_i are positive unit vectors such that $y_i^{\mathrm{T}} x_i = 1$, and B, C are nonnegative matrices of appropriate size.

We may write

$$PAP^{\mathrm{T}} = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} = FG,$$

say. This gives a full-rank factorization of PAP^{T} because if $r = \operatorname{rank} PAP^{\mathrm{T}} = \operatorname{rank} X = \operatorname{rank} F$ and $r = \operatorname{rank} PAP^{\mathrm{T}} = \operatorname{rank} Y = \operatorname{rank} TY = \operatorname{rank} G$. Now, we know that if GF is invertible, then $A^{\#}$ exists. So, consider the following:

$$GF = \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} = TYX = T$$

(because $YX = I_r$ the $r \times r$ identity matrix). By our hypothesis, T is invertible. Thus, GF is invertible.

Therefore we have

$$\begin{aligned} PA^{\#}P^{\mathrm{T}} &= F(GF)^{-2}G \\ &= \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \left(\begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \right)^{-2} \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} (T)^{-2} \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} T^{-1}Y & T^{-1}YB & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} XT^{-1}Y & XT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXT^{-1}Y & CXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This in turn yields that

$$PAA^{\#}P^{\mathrm{T}} = PAP^{\mathrm{T}}PA^{\#}P^{\mathrm{T}} = FGF(GF)^{-2}G = F(GF)^{-1}G$$

$$= \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \left(\begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} (T)^{-1} \begin{bmatrix} TY & TYB & 0 & 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} Y & YB & 0 & 0 \end{bmatrix} \geqslant 0.$$

So, we have shown that $PAA^{\#}P^{\mathrm{T}} \geqslant 0$, and since P is a permutation matrix, we obtain $AA^{\#} \geqslant 0$. This proves (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i): Obvious. □

The following example illustrates that the class of nonnegative matrices A with $AA^{\#}$ nonnegative is properly contained in the class of nonnegative A with $A^{\#}$ nonnegative.

Example 2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We can easily verify that $A^{\#} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ngeqslant 0$.

But clearly, $AA^{\#} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geqslant 0$.

We next obtain the following corollary, analogous to the result that if there exists a nonnegative $A^{(1,5)}$ then $A^{\#} = A^{(1,5)}$.

Corollary 3. *If A is a nonnegative matrix such that $AA^{(1,5)} \geqslant 0$, then $AA^{\#} = AA^{(1,5)}$.*

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Proof. By Theorem 1, there exists a permutation matrix such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

as described in the statement, and

$$PAA^{(1,5)}P^{\mathrm{T}} = \begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

as shown in the proof.

Also, by Theorem 1

$$PA^{\sharp}P^{\mathrm{T}} = \begin{bmatrix} XT^{-1}Y & XT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXT^{-1}Y & CXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, we calculate

$$\begin{aligned} &PAP^{\mathrm{T}}PA^{\sharp}P^{\mathrm{T}} \\ &= \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} XT^{-1}Y & XT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXT^{-1}Y & CXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} XTYXT^{-1}Y & XTYXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTYXT^{-1}Y & CXTYXT^{-1}YB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} XY & XYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXY & CXYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, we have shown that $PAA^{(1,5)}P^{\mathrm{T}} = PAA^{\sharp}P^{\mathrm{T}}$, and because P is a permutation matrix, we have that $AA^{\sharp} = AA^{(1,5)}$, proving the theorem. □

Now, if in addition we assume that $A^{\sharp}$ is nonnegative, then we obtain the previous characterization of nonnegative matrices A with $A^{\sharp} \geqslant 0$ [7, Corollary 4.3, p. 111].

Corollary 4. *If A is a nonnegative matrix of* rank r *such that $A^\sharp \geqslant 0$, then there exists a permutation matrix P such that*

$$PAP^{\mathrm{T}} = \begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where B, C are nonnegative matrices of appropriate size, the diagonal blocks are square matrices and J is a direct sum of the following types (not necessarily both):

(I) *βxy^{T}, $\beta > 0$, x, y are positive unit vectors of the same size and $y^{\mathrm{T}}x = 1$.*

(II)
$$\begin{bmatrix} 0 & \beta_{12}x_1x_2^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2x_3^{\mathrm{T}} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \beta_{d-1}x_{d-1}x_d^{\mathrm{T}} \\ \beta_{d1}x_dx_1^{\mathrm{T}} & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

with $\beta_{ij} > 0$; x_i, y_i are positive unit vectors, x_i, y_i are of the same size with $y_i^{\mathrm{T}}x_i = 1$, $x_i, y_j, i \neq j$ are not necessarily the same size.

Proof. Because we have that A and $A^\sharp$ are nonnegative, $AA^\sharp$ is nonnegative, and so by Theorem 1, we have a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} XTY & XTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CXTY & CXTYB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal blocks are square, T is a nonnegative invertible matrix,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix}, \quad Y = \begin{bmatrix} y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & y_2^{\mathrm{T}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^{\mathrm{T}} \end{bmatrix},$$

x_i and y_i are positive unit vectors such that $y_i^{\mathrm{T}}x_i = 1$, and B, C are nonnegative matrices of appropriate size.

So, what we need to do is show that $XTY = J$ as described in our hypothesis.

Now, $A^\sharp$ nonnegative implies that $XT^{-1}Y \geqslant 0$. But then, we have $YXT^{-1}YX \geqslant 0$. This implies that T^{-1} is nonnegative because $YX = I$.

Now T^{-1} nonnegative trivially implies that T has one and only one nonzero entry in each row and each column, which is the "same" as a permutation matrix with the exception that the nonzero entries need not be one. Then, since every permutation in S_n, the symmetric group of n elements, can be expressed as a product of disjoint cycles, it follows that there exists a permutation matrix P such that PTP^{-1} is a direct sum of Type (I) and Type (II) matrices. Furthermore, because

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{bmatrix}, \quad Y = \begin{bmatrix} y_1^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & y_2^{\mathrm{T}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_r^{\mathrm{T}} \end{bmatrix},$$

it follows that there exists a permutation matrix Q such that $QXYQ^{-1}$ is a direct sum of Type (I) and Type (II) matrices, proving our result. □

3. Nonnegative matrices A with $A - A^2 \geqslant 0$

Since $A^{\#}$ is a polynomial given by $A(q(A))^2$, where $\lambda(1 - \lambda(q(\lambda))$ is the minimal polynomial of A and $AA^{\#} = A(q(A))$, it is natural to ask a more general question as to when $A(p(A))$ is nonnegative, where $p(X)$ is any polynomial. In this section we consider a special case $p(X) = I - X$ and ask as to when $A - A^2 \geqslant 0$. This result is of independent interest because it generalizes Flor's theorem on nonnegative idempotents. We show that if A is an irreducible matrix with $\rho(A) = 1$, then $A - A^2 \geqslant 0$ implies $A = A^2$. Clearly, in general, A need not be idempotent if $A - A^2 \geqslant 0$. We are unable to give a complete characterization for reducible matrices. Theorem 7 gives a necessary condition for any nonnegative matrix A with $A - A^2 \geqslant 0$. We close with an example that shows that the conditions obtained in Theorem 7 are not sufficient. The next known lemma is part of the folklore of irreducible matrices.

Lemma 5. *If $A \geqslant 0$ is irreducible, $B \geqslant 0$, then $AB = 0$ or $BA = 0$ implies that $B = 0$.*

We first prove the following lemma.

Lemma 6. *Let $A \geqslant 0$ be an irreducible matrix such that $\rho(A) = 1$. Then $A \geqslant A^2$ if and only if $A = A^2$.*

Proof. Let $B = I - A$. Because $\rho(A) = 1$ we know that B is an M-matrix.

Also, we know that because A is nonnegative, $1 \in \operatorname{spec}(A)$ and therefore, $0 \in \operatorname{spec}(B)$. This implies that B is singular. Clearly, A irreducible implies that B is irreducible. So, by [2, Theorem 4.16, p. 156], B is almost monotone, i.e., $Bx \geqslant 0 \Rightarrow Bx = 0$.

Let $X^{(i)}$ denote the ith column of the matrix X. Set $x = A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = A^{(1)}$.

Then,

$$Bx = (I - A)A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = (A - A^2)\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = (A - A^2)^{(1)} \geqslant 0$$

by hypothesis. But then, we must have that $(A - A^2)^{(1)} = 0$. Similarly, we can show that $(A - A^2)^{(i)} = 0$, for all i. Thus, we have just shown that $A - A^2 = 0 \Rightarrow A = A^2$.

The converse is clear. □

We now prove our main result, giving a necessary condition for nonnegative matrices A with $A - A^2 \geqslant 0$.

Theorem 7. *Let $A \geqslant 0$ with $\rho(A) = 1$. If $A - A^2 \geqslant 0$ then there exists a permutation matrix P such that*

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix},$$

where each A_{ii} is irreducible or a 1×1 zero matrix. If for all i, $A_{ii} \neq 0$, then there exists an i such that $A_{ii} = A_{ii}^2$ and $A_{ij} = 0$ for $j < i$, $A_{ki} = 0$ for $r \geqslant k > i$. The latter statement holds for each A_{ii} with $A_{ii} = A_{ii}^2$.

Proof. By the Frobenius normal form there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix},$$

where each A_{ii} is irreducible or a 1×1 zero matrix. If there exists a k such that $A_{kk} = 0$, then we are done, so suppose that for each i, $A_{ii} \neq 0$.

We proceed by induction on r. For $r = 2$, let $PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$. Then since $\rho(A) = 1$, either A_{11} or A_{22} has the same spectral radius.

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Now, we know that $A - A^2 \geqslant 0$ and so

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \geqslant 0.$$

This implies that

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{11}^2 & 0 \\ A_{21}A_{11} + A_{22}A_{21} & A_{22}^2 \end{bmatrix} \geqslant 0,$$

which in turn yields

$$\begin{bmatrix} A_{11} - A_{11}^2 & 0 \\ A_{21} - A_{21}A_{11} - A_{22}A_{21} & A_{22} - A_{22}^2 \end{bmatrix} \geqslant 0.$$

So, finally we have that

$$A_{ii} \geqslant A_{ii}^2 \quad \text{and} \quad A_{21} - A_{21}A_{11} - A_{22}A_{21} \geqslant 0.$$

Now, by Lemma 6, either A_{11} or A_{22} is idempotent.

Without loss of generality, suppose that A_{11} is idempotent. Consider the inequality

$$A_{21} - A_{21}A_{11} - A_{22}A_{21} \geqslant 0.$$

We then multiply this inequality by A_{11} on the right to obtain

$$A_{21}A_{11} - A_{21}A_{11}^2 - A_{22}A_{21}A_{11} \geqslant 0.$$

Then, because A_{11} is idempotent, we have

$$A_{21}A_{11} - A_{21}A_{11} - A_{22}A_{21}A_{11} \geqslant 0,$$

which implies that

$$-A_{22}A_{21}A_{11} \geqslant 0.$$

And so, $A_{22}A_{21}A_{11} = 0$ because all the matrices are nonnegative.

But, we know that A_{22} and A_{11} are irreducible. So using Lemma 5, we conclude that $A_{21} = 0$.

If A_{22} is idempotent, we would proceed as above and get the same conclusion, proving the result for $r = 2$.

Now suppose that the result holds for all positive integers $< r$. We will show that it is true for r.

Assume

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix}.$$

Then $A - A^2 \geqslant 0$ yields that $A_{ii} \geqslant A_{ii}^2$ for all r. Also, at least one of the A_{ii}'s has a spectral radius equal to 1. Therefore, by Lemma 6 we have some i such that $A_{ii} = A_{ii}^2$. If $i \neq r$, then consider the matrix

$$B = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{r-1,1} & A_{r-1,2} & \cdots & A_{r-1,r-1} \end{bmatrix}.$$

By the induction hypothesis, we have that $A_{ij} = 0$ for $j < i$, $A_{ki} = 0$ for $r - 1 \geqslant k > i$.

We need only show that $A_{ri} = 0$. As before, $A - A^2 \geqslant 0$ yields that

$$A_{ri} - (A_{ri}A_{ii} + A_{r,i+1}A_{i+1,i} + \cdots + A_{rr}A_{ri}) \geqslant 0.$$

So now, using $A_{ij} = 0$ for $j < i$, $A_{ki} = 0$ for $r - 1 \geqslant k > i$, we obtain

$$A_{ri} - A_{ri}A_{ii} - A_{rr}A_{ri} \geqslant 0.$$

Multiply on the right by A_{ii} to obtain

$$A_{ri}A_{ii} - A_{ri}A_{ii}A_{ii} - A_{rr}A_{ri}A_{ii} \geqslant 0,$$

which implies that

$$A_{ri}A_{ii} - A_{ri}A_{ii}^2 - A_{rr}A_{ri}A_{ii} \geqslant 0.$$

This yields $-A_{rr}A_{ri}A_{ii} \geqslant 0$.

Thus $A_{rr}A_{ri}A_{ii} = 0$. Also, we know that A_{rr} and A_{ii} are irreducible, and so $A_{ri} = 0$.

If $r = i$, we use

$$B = \begin{bmatrix} A_{22} & 0 & \cdots & 0 \\ A_{32} & A_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{r2} & A_{r3} & \cdots & A_{rr} \end{bmatrix}$$

and similarly show that $A_{r1} = 0$. □

The following example illustrates the importance of all of the matrices on the diagonal of the Frobenius normal form being nonzero.

Example 8. Let $A = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$. Clearly, $A = A^2$ and $\rho(A) = 1$. For this matrix, we have that $A_{22}^2 = A_{22}$, yet $A_{21} = 2 > 0$.

We now provide an example showing that the condition obtained in the above theorem is not sufficient.

Example 9. Let

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & \frac{1}{2} & 0 \\ \frac{1}{10} & 0 & \frac{1}{10} & \frac{1}{2} \end{bmatrix}.$$

Clearly, the spectral radius is one. Also, the irreducible blocks on the diagonal are $\left[\frac{1}{2}\right]$, $[1]$, $\left[\frac{1}{2}\right]$, and $\left[\frac{1}{2}\right]$. The only idempotent block is $[1]$, and it satisfies the conclusion of the theorem that all entries below and to the left of this block are zero. However,

$$A - A^2 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ -\frac{1}{100} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

is not nonnegative.

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