Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 275 (2004) 856–858
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

# Semilocal CS matrix rings of order > 1 over group algebras of solvable groups are selfinjective

K.I. Beidar,[a] S.K. Jain,[b,*] Pramod Kanwar,[c] and J.B. Srivastava[d]

[a] *Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan*
[b] *Department of Mathematics, Ohio University, Athens, OH 45701, USA*
[c] *Department of Mathematics, Ohio University–Zanesville, Zanesville, OH 43701, USA*
[d] *Department of Mathematics, Indian Institute of Technology, Delhi 110016, India*

Received 30 April 2003

Communicated by Kent R. Fuller

*Keywords:* CS-rings; finitely $\sum$-CS modules; Semilocal rings; Locally finite groups; Solvable groups; Linear groups

The following theorem is a sort of an addendum or a sequel to our earlier paper [1]. This result generalizes Theorem 4.3 in [1] to group algebras of solvable groups. For definitions and terminology the reader is referred to [1].

**Theorem 0.1.** *Let $K$ be a field and $G$ be a group. Suppose one of the following conditions is satisfied.*

(a) *$G$ is a locally finite group.*
(b) *The group algebra $KG$ is semilocal and $G$ is either a solvable group or a linear group.*

*Then the following are equivalent.*

(1) *$M_n(KG)$, $n > 1$, is a right CS-ring.*
(2) *$M_2(KG)$ is a right CS-ring.*
(3) *$KG$ is right selfinjective.*

* Corresponding author.
*E-mail addresses:* kbeidar@yahoo.com (K.I. Beidar), jain@math.ohiou.edu (S.K. Jain), pkanwar@math.ohiou.edu (P. Kanwar), jbsrivas@maths.iitd.ernet.in (J.B. Srivastava).

0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2003.06.008

(4) *G is finite.*

**Proof.** (1) ⇒ (2) is well-known [2, Lemmas 7.1 and 12.8].

(2) ⇒ (3). First assume that $G$ is a locally finite group. Write $R = M_2(KG)$. By [4, Corollary 8.4], it is enough to prove that $R$ is right continuous, that is, any right ideal of $R$ which is isomorphic to a direct summand of $R$ is itself a direct summand of $R$. Observe that for any two elements $a$ and $b$ of $R = M_2(KG)$ there exists a right selfinjective subring $T$ containing $a$, $b$, and the unity of $R$. To see this let $a$ and $b$ be any two elements of $R$ and let $H$ be the subgroup generated by the supports of the entries of $a$ and $b$. Since $G$ is locally finite, $H$ must be finite. Hence $KH$ is right selfinjective [3, Theorem 2.8, p. 79]. So $T = M_2(KH)$ is the desired right selfinjective subring of $R$ containing $a$, $b$, and the unity of $R$. To prove $R$ is right continuous, let $I$ be a right ideal of $R$ and let $I \simeq eR$ for some idempotent $e \in R$. Suppose $\alpha : I_R \to eR_R$ is an isomorphism. Let $a = \alpha^{-1}(e)$. Then, as shown above, there exists a right selfinjective subring, say $S$, containing unity of $R$ such that $a, e \in S$. Obviously $\alpha$ induces an isomorphism $aS_S \to eS_S$. Since $S$ is right selfinjective, $eS$ is an injective right $S$-module. Therefore $aS$ is an injective right $S$-module. Thus $aS = vS$ for some idempotent $v \in S$. Since $S$ contains the unity of $R$, it follows that $aR = vR$. Consequently $R$ is right continuous.

Now let $KG$ be semilocal and $G$ be either solvable or linear. By [3, Section 3, p. 322]

$$\begin{aligned} J(KG) &= N^*(KG) \\ &= \{\alpha \in KG \mid \alpha S \text{ is nilpotent for every finitely generated subring } S \text{ of } KG\}. \end{aligned}$$

In particular, $J(KG)$ is nil. By [3, Theorem 1.5, p. 409], $G$ is locally finite. But then by what we have proved above, $KG$ is right selfinjective.

(3) ⇒ (1) follows from the fact that the matrix ring over a right selfinjective ring is again right selfinjective and right selfinjective rings are right CS-rings.

The equivalence of (3) and (4) is well-known [3, Theorem 2.8, p. 79]. □

**Remark 1.** The above proof shows that if $R$ is a right CS-ring with unity 1 such that any two elements $a, b \in R$ are contained in a right selfinjective subring $S$ having the same unity 1 then $R$ is right continuous. In particular, if $R = M_2(T)$ for some ring $T$ then $R$ (and hence $T$) is right selfinjective.

**Remark 2.** It follows from Remark 1 that a group algebra of a locally finite group is right CS if and only if it is right continuous.

**Remark 3.** Theorem 0.1 is not true if the order of the matrix ring is not greater than 1. For example, the group ring of an infinite locally finite $p$-group over a field of characteristic $p$ is a local right CS-ring but is not right selfinjective.

## Acknowledgment

The authors thank the referee for his helpful suggestions.

## References


[1] K.I. Beidar, S.K. Jain, P. Kanwar, J.B. Srivastava, CS matrix rings over local rings, J. Algebra 264 (1) (2003) 251–261.

[2] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Pitman, London, 1994.

[3] D.S. Passman, The Algebraic Structure of Group Rings, Wiley, New York, 1977.

[4] Y. Utumi, On continuous and selfinjective rings, Trans. Amer. Math. Soc. 118 (1965) 158–173.