

## Prime Goldie Rings of Uniform Dimension at Least Two and with All One-Sided Ideals CS Are Semihereditary

Dinh Van Huynh,\* S. K. Jain, and S. R. López-Permouth

Department of Mathematics, Ohio University,  
Athens, Ohio, USA

Throughout this note, all rings are assumed to be associative and to have an identity. Likewise, all modules are supposed to be unitary.

A right  $R$ -module  $M_R$  is called a CS module (or an extending module) if every submodule of  $M$  is essential in a direct summand of  $M$ . A ring  $R$  is called right CS if  $R_R$  is a CS module. While the class of CS modules contains all injective (indeed quasi-continuous) modules, it also contains all uniform modules as well as all semisimple modules. With such a

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\*Correspondence: Dinh Van Huynh, Department of Mathematics, Ohio University, Athens, OH 45701, USA; E-mail: huynh@ohio.edu.



wide range, it is somewhat surprising that one can get much structure from the assumption that certain modules satisfy CS. The tremendous growth of the theory of CS modules in recent years is documented in Dung et al. (1994), which is a good reference for the foundational results on the subject.

Rings whose right ideals are quasi-injective or quasi-continuous have been studied by a large number of authors (c.f. Byrd, 1979; Ivanov, 1972, 1996; Jain et al., 1969, 1999, etc.) This paper is about rings for which finitely generated right ideals are CS. It must be mentioned that the structure of rings with all right ideals CS has not been determined. There exist right CS rings not all of whose right ideals are CS (see Birkenmeier et al., 2000). In this note we show that for semiprime right Goldie rings with uniform dimension at least 2, the condition that all finitely generated right ideals are CS is equivalent to that of being semihereditary. Consequently, for simple right CS right QI rings  $R$  with  $u\text{-dim}(R_R)$  at least 2, Boyle's conjecture that every simple right QI-ring is hereditary is equivalent to the statement that all finitely generated right (or left) ideals of  $R$  are CS.

Note that if a ring  $R$  is uniform as a right module then all right ideals of  $R$  satisfy CS. However, there are right Ore domains that are not right semihereditary. In this paper we consider only rings with uniform dimension at least 2.

**Theorem 1.** *Let  $R$  be a prime right Goldie ring with  $u\text{-dim}(R_R) \geq 2$ . Then the following conditions are equivalent:*

- (a) *Every finitely generated right ideal of  $R$  is CS.*
- (b) *Every finitely generated left ideal of  $R$  is CS.*
- (c)  *$R$  is left Goldie, right semihereditary.*
- (d)  *$R$  is left Goldie and left semihereditary.*

*Proof.* (a)  $\Rightarrow$  (c). By Goodearl and Warfield (1989),  $R$  is left Goldie. Since  $R$  is right CS,  $R = R_1 \oplus \cdots \oplus R_n$ , where each  $R_i$  is uniform (see Dung et al., 1994, 8.2). Let  $U$  be a finitely generated uniform right ideal of  $R$ . As  $R$  is a prime right Goldie ring,  $R_i$  is subsomorphic to  $R_j$  (see Goodearl and Warfield, 1989, Lemma 6.22). As  $n \geq 2$ , there is an  $R_j$  with  $R_j \cap U = 0$ . Let  $\varphi$  be a monomorphism of  $R_i$  into  $R_j$ . Then being a finitely generated right ideal of  $R$ ,  $U \oplus \varphi(R_i)$  is CS. Consequently,  $U \oplus R_i$  is CS for each  $i = 1, 2, \dots, n$ .

We next prove, by induction on  $k$ , that  $R_R^k$  has the extending property for uniform submodules. We denote this condition by  $(1-C_1)$ . By (a) this is true for  $k = 1$ . Assume that  $R_R^k$  has  $(1-C_1)$  for a fixed  $k \in \mathbb{N}$ .

Let  $U$  be a uniform closed submodule of  $M_1 = R_R^k \oplus V_1$  with  $V_1 = R_1$ . If  $U \cap V_1 \neq 0$ , then  $U = V_1$  because of the uniqueness of essential closures in nonsingular modules. Consider  $V_1 \cap U = 0$ , and set  $U_1 = V_1 \oplus U$ . By modularity,  $U_1 = V_1 \oplus U'$  where  $U' = R_R^k \cap U_1$ . There is a uniform direct summand  $U^*$  of  $R_R^k$  containing  $U'$ . Hence  $V_1 \oplus U^*$  is a direct summand of  $M_1$ . Obviously,  $U^*_R$  is finitely generated, and embeds in any  $R_i$ , say in  $R_2$ . Hence, by the hypothesis,  $V_1 \oplus U^*$  is CS. It follows that  $U$  is a direct summand of  $V_1 \oplus U^*$ , and hence of  $M_1$ . This shows that  $M_1$  has the property  $(1-C_1)$ . Repeating this argument yields that  $M_1 \oplus V_2$  is  $(1-C_1)$  where  $V_2 = R_2$ . Continuing in this way, we obtain the conclusion that  $R_R^{k-1} (= R_R^k \oplus R_1 \oplus \dots \oplus R_n)$  has  $(1-C_1)$ .

Now let  $C$  be a closed submodule of  $M = R_R^m$ ,  $m \in \mathbb{N}$ , and let  $W_1$  be a closed uniform submodule of  $C$ , which is also closed in  $M$ . Hence  $M = W_1 \oplus T_1$  for some submodule  $T_1$  of  $M$ . By modularity,  $C = W_1 \oplus C_1$  where  $C_1 = T_1 \cap C$ . If  $C_1$  is nonzero then, being a closed submodule of  $C$ ,  $C_1$  contains a closed uniform submodule  $W_2$ . As a direct summand of  $M$ ,  $T_1$  has also  $(1-C_1)$ . Hence  $T_1 = W_2 \oplus T_2$ , and therefore,  $C = W_1 \oplus W_2 \oplus C_3$  with  $C_3 = T_2 \cap C$ . Since  $C$  has finite uniform dimension, continuing in this way we finally reach a direct decomposition  $C = W_1 \oplus \dots \oplus W_k$ , where each  $W_i$  is uniform, and  $C$  is a direct summand of  $M$ . This proves that  $M$  is CS. It follows that  $R$  is right semihereditary by Dung et al. (1994, 12.17).

(c)  $\Leftrightarrow$  (d) by Dung et al. (1994, 12.18).

(d)  $\Rightarrow$  (b). By (d)  $R$  is left semihereditary, right and left Goldie. By Dung et al. (1994, 12.18), for each  $t \in \mathbb{N}$ ,  $R^t$  is CS as a left  $R$ -module. Let  $A$  be a finitely generated left ideal of  $R$ . Then there is an epimorphism  $\varphi : {}_R R^m \rightarrow {}_R A$ . Since  ${}_R A$  is nonsingular,  $\ker \varphi$  is closed in  ${}_R R^m$ . Hence  ${}_R R^m = \ker \varphi \oplus B$  for some submodule  $B$  of  ${}_R R^m$ . Since  $B \cong A$ ,  ${}_R A$  is CS, proving (b).

(b)  $\Rightarrow$  (d) is similar to (a)  $\Rightarrow$  (c), and (c)  $\Rightarrow$  (a) is similar to (d)  $\Rightarrow$  (b). □

Notice that by a result of Chatters that decomposes every noetherian hereditary ring as a (finite) direct sum of rings that are prime or artinian (c.f. Faith, 1976a, 20.30), Theorem 1 implies that every right or left ideal of a semiprime noetherian hereditary ring is CS. However, this is not true without the assumption of semiprimeness; consider, for example, the artinian hereditary ring  $\begin{bmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$  with a field  $F$ , that



has a closed submodule  $\left\{ \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \mid t \in F \right\}$  which is clearly not a direct summand.

**Theorem 2.** *For a semiprime right Goldie ring  $R$ , the following conditions are equivalent:*

- (i) *Every finitely generated right ideal of  $R$  is CS.*
- (ii)  *$R$  has a ring-direct decomposition  $R = R_1 \oplus \cdots \oplus R_n$ , where each  $R_i$  is either a right Ore domain, or a prime right and left Goldie, right and left semihereditary ring.*

*Proof.* (i)  $\Rightarrow$  (ii). Since  $R$  is right CS,  $R_R$  is a direct sum of uniform right ideals  $U_i$ . Let  $[U_k]$  be the direct sum of all such direct summands  $U_i$  which are subisomorphic to  $U_k$ . Then each  $R_k = [U_k]$  is a prime ideal of  $R$  (see McConnell and Robson, 1987, 3.3 Corollary). Therefore,  $R = R_1 \oplus \cdots \oplus R_n$  where each  $R_i$  is either a right Ore domain, or a prime right Goldie right CS ring. If  $R_i$  is not a domain, it is then a right and left Goldie, right and left semihereditary ring by Theorem 1.

(ii)  $\Rightarrow$  (i) is clear by Theorem 1. □

**Corollary 3.** *For a semiprime right and left noetherian ring  $R$  the following conditions are equivalent:*

- (1) *Every right ideal of  $R$  is CS.*
- (2)  *$R = R_1 \oplus \cdots \oplus R_n$  where each  $R_i$  is either a domain or a prime right and left hereditary ring.*

*Proof.* (1)  $\Rightarrow$  (2). By Theorem 2,  $R = R_1 \oplus \cdots \oplus R_n$  where each  $R_i$  is a domain, or a prime right and left semihereditary ring. In the latter case  $R_i$  will be a right and left hereditary ring. Note that some  $R_i$  can be simple artinian rings.

(2)  $\Rightarrow$  (1) is clear by Theorem 2. □

A ring  $R$  is called a right QI ring if every quasi-injective right  $R$ -module is injective. A conjecture of A. Boyle says that any right QI ring is right hereditary (see Faith, 1976a). While this conjecture remains unproved, we will give here a sufficient condition for a right QI ring to be right hereditary.

By Faith (1976b, Theorem 2), a right QI ring is the ring direct sum of simple right noetherian right V-rings. Hence we may restrict ourselves to simple right QI rings.

**Corollary 4.** For a simple right QI ring  $R$  with  $u\text{-dim}(R_R) \geq 2$ , the following conditions are equivalent:

- (a) Every right ideal of  $R$  is CS.
- (b) Every finitely generated left ideal of  $R$  is CS.
- (c)  $R$  is right hereditary, left Goldie.
- (d)  $R$  is left semihereditary, left Goldie.

*Proof.* Since every right QI ring is right noetherian (see Faith, 1976a, Proposition 20.4B), the proof follows from Theorem 1.  $\square$

If the ring  $R$  in Corollary 4 were left noetherian, then  $R$  would be left hereditary. In this case,  $R$  would be Morita equivalent to a right and left PCI domain. A right PCI domain is a simple right hereditary right noetherian domain  $D$  such that every singular right  $D$ -module is semisimple and injective (see Damiano, 1979; Faith, 1973). We note that by another result of Faith (1973, Theorem 22), a left Ore right PCI domain is left PCI, hence left noetherian and left hereditary.

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