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Columns of uniform color in a rectangular array with rows having cyclically repeated color patterns

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Abstract

Given positive integers $p_1, \ldots, p_n$, consider a rectangular array of balls of m different colors, with the ith row containing, in repeated succession, p_i balls of each color. This paper initiates the problem of finding the number of columns having balls of the same color, and obtains the solution in various cases.

MSC: Primary 05A15; 05A19

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1. Introduction

Let $p_1, \ldots, p_n$ be pairwise coprime positive integers (hereafter simply called coprimes), and $P = p_1 \cdots p_n$. Let m be an integer > 1. Suppose we have mnP balls in m different colors, denoted $0, 1, \ldots, m-1$, there being nP balls of each color. These are arranged in a rectangular array with n rows, each row having P balls of each color in the following manner. The ith row contains in cyclic succession p_i balls of color 0, p_i balls of color $1, \ldots, p_i$ balls of color $m-1$. We consider the problem of finding the number of columns having balls of the same color, hereafter referred to as *matching columns*. These matching columns may occur in isolation or in clusters. By a *matching block* of width k we shall mean a set of k consecutive matching columns in the same color which are not preceded or followed immediately by matching columns of the same color.

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For example, the diagram below shows two rows of white and black balls, with $p_1=3$, $p_2=5$. Each row has 15 white and 15 black balls. By direct observation, we find that there are 16 matching columns, distributed in 4 matching blocks of width 1, and 4 matching blocks of width 3.

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Mathematically, the rectangular array of colored balls described above can be represented by an $n\times mP$ matrix A in which the ith row has in cyclic succession p_i 0's, p_i 1's, ..., p_i $(m-1)$'s. A column of A is a matching column if all entries in it are the same. The array of balls shown in the above diagram is represented by the matrix

$$\begin{bmatrix} 000111000111000111000111000111 \\ 000001111100000111110000011111 \end{bmatrix}.$$

In Section 2 we consider the case $m=2$, and obtain formulas for the number of matching columns, the total number of matching blocks and the number of matching blocks of any given width.

In Section 3 we consider the general case of m colors, and assume that all rows start with the same color. In Theorem 6, we assume that $p_1,\dots,p_n$ are all congruent to one another modulo m. (This corresponds to the condition that $p_1,\dots,p_n$ are all odd in the earlier case $m=2$.) As before, we find formulas for the number of matching columns, etc. Theorem 7 considers the special case $n=2$, with p_1-p_2 relatively prime to m.

In the final section, we derive from Theorem 6 two identities in $p_1,\dots,p_n$, which are of independent interest. These lead to two recursion relations (Theorem 9) for the sums

$$B_j(p)=s^j+(s+m)^j+\cdots+(p-m)^j,$$

where $p\equiv s\pmod m$, $1\leqslant s\leqslant m-1$.

The motivation for studying this problem arose from a question asked by a bio-mathematician working on the multiple sequence problem that deals with finding, for given k sequences of characters from a fixed alphabet, an alignment with optimal score according to a given scoring scheme. This problem is of crucial importance in molecular biology [1].

2. Matching columns and blocks with 2 colors

In this section we assume $m=2$ and allow some rows to begin with 0's and others with 1's. Without loss of generality, we may assume that the rows starting with 0's are at the top. So we have an $n\times 2P$ matrix A in which the ith row contains alternately p_i 0's and p_i 1's. It is clear that, because of symmetry, a column x in A is matching if and only if the column $2P+1-x$ is matching. Therefore, if we partition A into two submatrices having P columns each, the two submatrices have the same number of matching columns. So we need to consider a matrix A of size $n \times P$ only. We shall obtain a formula for the total number of matching columns and the total number of matching blocks in A as well as the number of matching blocks of any given width k. It is obvious that the width of a matching block in A cannot exceed p, where $p = \min(p_1, \dots, p_n)$.

Let us write $S_r(p_1, \dots, p_n)$ to denote the sum of all products of $p_1, \dots, p_n$, taken r at a time, with $S_0(p_1, \dots, p_n) = 1$. So $S_n(p_1, \dots, p_n) = p_1 \cdots p_n$, and, for $0 < r < n$, $S_r(p_1, \dots, p_n)$ is the sum of $\binom{n}{r}$ terms of the type $p_1 \cdots p_r$. As usual, for any real number x, $[x]$ denotes the greatest integer less than or equal to x.

We first consider the case where $p_1, \dots, p_n$ are odd coprimes.

Theorem 1. *Let $p_1, \dots, p_n$ be odd coprimes, and $P = p_1 \cdots p_n$. Let $p = \min(p_1, \dots, p_n)$. Let A be an $n \times P$ matrix in which the ith row has alternately p_i 0's and p_i 1's, and suppose that the first t rows start with 0's, and the remaining $n - t$ rows start with 1's. Then*

(a) *The number M of matching columns in A is*

$$M = \frac{1}{2^{n-1}} \left\{ p_1 \cdots p_n + (-1)^{n-t} \sum_{r=1}^{[n/2]} S_{n-2r}(p'_1, \dots, p'_n) \right\},$$

where $p'_i = p_i$ for $i = 1, \dots, t$, and $p'_i = -p_i$ for $i = t+1, \dots, n$.

(b) *The number N of matching blocks in A is*

$$N = \frac{1}{2^{n-1}} \left\{ (-1)^{n-t} \sum_{r=1}^{[n/2]} S_{n-2r}(p'_1, \dots, p'_n) - \sum_{r=1}^{n} (-1)^r S_{n-r}(p_1, \dots, p_n) \right\}.$$

(c) *The number of matching blocks of width p is*

$$N(p) = \begin{cases} \frac{1}{2^n} \prod_{i=1}^{t} (p_i - p + 2) \prod_{i=t+1}^{n} (p_i - p) & \text{if } p \in \{p_1, \dots, p_t\}, \\ \frac{1}{2^n} \prod_{i=1}^{t} (p_i - p) \prod_{i=t+1}^{n} (p_i - p + 2) & \text{if } p \in \{p_{t+1}, \dots, p_n\}. \end{cases}$$

(d) *For every odd positive integer $k < p$, the number of matching blocks of width k is*

$$N(k) = 2 \left\{ \sum_{r=1}^{[t/2]} S_{t-2r}(q_1, \dots, q_t) q_{t+1} \cdots q_n + q_1 \cdots q_t \sum_{r=1}^{[(n-t)/2]} S_{n-t-2r}(q_{t+1}, \dots, q_n) \right\},$$

where

$$q_i = \tfrac{1}{2}(p_i - k) \quad \text{for each } i = 1, \dots, n.$$

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(e) *For every even positive integer* $k < p$,

$$N(k) = 2\sum_{r=1}^{n} S_{n-r}(q'_1,\dots,q'_n)$$

$$-2\left\{q'_1\cdots q'_t\sum_{r=1}^{n-t} S_{n-t-r}(q'_{t+1},\dots,q'_n) + q'_{t+1}\cdots q'_n\sum_{r=1}^{t} S_{t-r}(q'_1,\dots,q'_t)\right\},$$

where

$$q'_i = \tfrac{1}{2}(p_i - k - 1) \quad \textit{for each } i = 1,\dots,n.$$

Proof. Given any integer x, $1 \leqslant x \leqslant P$, let $b_i p_i$ be the largest multiple of p_i less than x, $i = 1,\dots,n$. Then it is clear from the definition of the matrix A that a_{ix}, the (i,x)th entry in A, is given by the following rule. For $i = 1,\dots,t$,

$$a_{ix} = \begin{cases} 0 & \text{if } b_i \text{ is even}, \\ 1 & \text{if } b_i \text{ is odd}. \end{cases}$$

But for $i = t+1,\dots,n$,

$$a_{ix} = \begin{cases} 1 & \text{if } b_i \text{ is even}, \\ 0 & \text{if } b_i \text{ is odd}. \end{cases}$$

Write $k_i = x - b_i p_i$. Then $1 \leqslant k_i \leqslant p_i$. Conversely, consider any n-tuple $(k_1,\dots,k_n)$ such that $1 \leqslant k_i \leqslant p_i$. By hypothesis, $p_1,\dots,p_n$ are coprimes. Hence, by Chinese remainder theorem, there is a unique x (lying between 1 and $p_1\cdots p_n$) such that

$$x \equiv k_i \pmod{p_i} \quad \text{for each } i = 1,\dots,n.$$

Hence $x = b_i p_i + k_i$ for some b_i $(i = 1,\dots,n)$. Thus, there is a one-to-one correspondence between the columns of the matrix A and the n-tuples $(k_1,\dots,k_n)$ with $1 \leqslant k_i \leqslant p_i$, given by

$$k_i = x - b_i p_i.$$

(a) Suppose the xth column in A is a matching column.Then either $a_{ix} = 0$ for all i, or $a_{ix} = 1$ for all i. Hence, either

$$b_1,\dots,b_t \text{ are all even} \quad \text{and} \quad b_{t+1},\dots,b_n \text{ all odd}$$

or

$$b_1,\dots,b_t \text{ are all odd} \quad \text{and} \quad b_{t+1},\dots,b_n \text{ all even}.$$

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Since $p_1, \ldots, p_n$ are all odd, it follows that either

(a) $k_1, \ldots, k_t$ are all odd and $k_{t+1}, \ldots, k_n$ are all even, or
(b) $k_1, \ldots, k_t$ are all even and $k_{t+1}, \ldots, k_n$ are all odd.

Conversely, if $k_1, \ldots, k_n$ satisfy this condition, the xth column is matching. Therefore, because of the one-to-one correspondence between the columns of A and the n-tuples $(k_1, \ldots, k_n)$, the number of matching columns is equal to the number of n-tuples $(k_1, \ldots, k_n)$ satisfying the following conditions:

(1) $1 \leqslant k_i \leqslant p_i$,
(2) either

$$k_1, \ldots, k_t \text{ are all odd} \quad \text{and} \quad k_{t+1}, \ldots, k_n \text{ are all even}$$

or

$$k_1, \ldots, k_t \text{ are all even} \quad \text{and} \quad k_{t+1}, \ldots, k_n \text{ are all odd}.$$

These n-tuples $(k_1, \ldots, k_n)$ are given by

$$k_i = \begin{cases} 1, 3, \ldots, p_i & (i = 1, \ldots, t) \\ 2, 4, \ldots, p_i - 1 & (i = t+1, \ldots, n) \end{cases}$$

or

$$k_i = \begin{cases} 2, 4, \ldots, p_i - 1 & (i = 1, \ldots, t), \\ 1, 3, \ldots, p_i & (i = t+1, \ldots, n). \end{cases}$$

Hence the number of n-tuples $(k_1, \ldots, k_n)$ satisfying conditions (1) and (2) is

$$M = \prod_{i=1}^{t} \left(\frac{p_i + 1}{2} \right) \prod_{i=t+1}^{n} \left(\frac{p_i - 1}{2} \right) + \prod_{i=1}^{t} \left(\frac{p_i - 1}{2} \right) \prod_{i=t+1}^{n} \left(\frac{p_i + 1}{2} \right).$$

On putting $p_i = p_i'$ for $i = 1, \ldots, t$, and $p_i = -p_i'$ for $i = t+1, \ldots, n$, we have

$$\begin{aligned} M &= \frac{(-1)^{n-t}}{2^n} \left\{ \prod_{i=1}^{n} (p_i' + 1) + \prod_{i=1}^{n} (p_i' - 1) \right\} \\ &= \frac{(-1)^{n-t}}{2^n} \left\{ \sum_{i=0}^{n} S_{n-i}(p_1', \ldots, p_n') + \sum_{i=0}^{n} (-1)^i S_{n-i}(p_1', \ldots, p_n') \right\} \\ &= \frac{(-1)^{n-t}}{2^{n-1}} \left\{ \sum_{r=0}^{[n/2]} S_{n-2r}(p_1', \ldots, p_n') \right\} \\ &= \frac{1}{2^{n-1}} \left\{ p_1 \cdots p_n + (-1)^{n-t} \sum_{r=1}^{[n/2]} S_{n-2r}(p_1', \ldots, p_n') \right\}. \end{aligned}$$

This is the number of matching columns in A.

(b) Consider an arbitrary matching block starting at column x. As before, let $b_i p_i$ be the largest multiple of $p_i < x$, and write $k_i = x - b_i p_i$. Then, arguing as before, $k_1, \ldots, k_n$ must satisfy the conditions (1) and (2) given above. Moreover, since column $x-1$ is not matching, we have

(3) $k_i = 1$ for some i.

Hence, the number of matching blocks is equal to the number of n-tuples $(k_1, \ldots, k_n)$ satisfying conditions (1)–(3).

The number of n-tuples $(k_1, \ldots, k_n)$ satisfying conditions (1) and (2), as already shown above, is

$$n_1 = \prod_{i=1}^{t}\left(\frac{p_i+1}{2}\right)\prod_{i=t+1}^{n}\left(\frac{p_i-1}{2}\right)+\prod_{i=1}^{t}\left(\frac{p_i-1}{2}\right)\prod_{i=t+1}^{n}\left(\frac{p_i+1}{2}\right).$$

Among these, the number of those that do not satisfy (3) is

$$n_2 = 2\prod_{i=1}^{n}\left(\frac{p_i-1}{2}\right).$$

Hence, the number of n-tuples $(k_1, \ldots, k_n)$ satisfying conditions (1)–(3) is

$$\begin{aligned}
N &= n_1 - n_2\\
&= \frac{1}{2^n}\left\{\prod_{i=1}^{t}(p_i+1)\prod_{i=t+1}^{n}(p_i-1)+\prod_{i=1}^{t}(p_i-1)\prod_{i=t+1}^{n}(p_i+1)-2\prod_{i=1}^{n}(p_i-1)\right\}\\
&= \frac{1}{2^{n-1}}\left\{(-1)^{n-t}\sum_{r=0}^{[n/2]} S_{n-2r}(p_1', \ldots, p_n') - \sum_{r=0}^{n}(-1^r)S_{n-r}(p_1, \ldots, p_n)\right\}\\
&= \frac{1}{2^{n-1}}\left\{(-1)^{n-t}\sum_{r=1}^{[n/2]} S_{n-2r}(p_1', \ldots, p_n') - \sum_{r=1}^{n}(-1^r)S_{n-r}(p_1, \ldots, p_n)\right\}.
\end{aligned}$$

This is the total number of matching blocks in A.

(c) Consider an arbitrary matching block of width k, consisting of columns $x, \ldots, x+k-1$. Let $b_i p_i$ be the largest multiple of $p_i < x$, and write $k_i = x - b_i p_i$ $(i = 1, \ldots, n)$. Then $1 \leqslant k_i \leqslant p_i - k + 1$ for each $i = 1, \ldots, n$. Hence k cannot exceed $p = \min(p_1, \ldots, p_n)$. Further, $k_1, \ldots, k_n$ satisfy the condition (2) above. Moreover, since the columns $x-1$ and $x+k$ are nonmatching, we must have $k_i = 1$ for some i, and also $k_i = p_i - k + 1$ for some i. Conversely, if $k_1, \ldots, k_n$ satisfy these conditions, then the columns $x, \ldots, x+k-1$ form a matching blocks. Therefore, because of the one-to-one correspondence between the columns of A and the n-tuples $(k_1, \ldots, k_n)$, the number of matching blocks of width k in A is equal to the number of n-tuples $(k_1, \ldots, k_n)$ satisfying the following conditions:

(1) $1 \leqslant k_i \leqslant p_i - k + 1$ $(i = 1, \ldots, n)$.

(2) Either

$$k_1,\dots,k_t \text{ are all odd and } k_{t+1},\dots,k_n \text{ are all even or}$$

$$k_1,\dots,k_t \text{ are all even and } k_{t+1},\dots,k_n \text{ are all odd.}$$

(3) $k_i = 1$ for some i.
(4) $k_i = p_i - k - 1$ for some i.

Consider first the case $k = p$. If $p \in \{p_1,\dots,p_t\}$, we may, without loss of generality, assume $p = p_1$. Then $k_1 = 1 = p_1 - k + 1$. Hence, $k_1,\dots,k_t$ are all odd and $k_{t+1},\dots,k_n$ are all even. Hence, the n-tuples $(k_1,\dots,k_n)$ satisfying the conditions (1)–(4) are given by $k_1 = 1$ and

$$k_i = \begin{cases} 1,3,\dots,p_i - p + 1 & (i = 2,\dots,t), \\ 2,4,\dots,p_i - p & (i = t+1,\dots,n). \end{cases}$$

The number of these n-tuples is

$$N(p) = \prod_{i=2}^{t} \left(\frac{p_i - p + 2}{2}\right) \prod_{i=t+1}^{n} \left(\frac{p_i - p}{2}\right)$$

$$= \frac{1}{2^n} \prod_{i=1}^{t} (p_i - p + 2) \prod_{i=t+1}^{n} (p_i - p).$$

This is the number of matching blocks of width p when $p \in \{p_1,\dots,p_t\}$. Similarly, if $p \in \{p_{t+1},\dots,p_n\}$ then

$$N(p) = \frac{1}{2^n} \prod_{i=1}^{t} (p_i - p) \prod_{i=t+1}^{n} (p_i - p + 2).$$

(d) Now suppose $k < p$. Consider first the case where k is odd. Then the n-tuples $(k_1,\dots,k_n)$ satisfying conditions (1) and (2) are given by

$$k_i = \begin{cases} 1,3,\dots,p_i - k + 1 & (i = 1,\dots,t), \\ 2,4,\dots,p_i - k & (i = t+1,\dots,n). \end{cases}$$

or

$$k_i = \begin{cases} 2,4,\dots,p_i - k & (i = 1,\dots,t), \\ 1,3,\dots,p_i - k + 1 & (i = t+1,\dots,n). \end{cases}$$

Hence, the number of n-tuples $(k_1,\dots,k_n)$ satisfying conditions (1) and (2) is

$$n_1 = \prod_{i=1}^{t} \left(\frac{p_i - k}{2} + 1\right) \prod_{i=t+1}^{n} \left(\frac{p_i - k}{2}\right) + \prod_{i=1}^{t} \left(\frac{p_i - k}{2}\right) \prod_{i=t+1}^{n} \left(\frac{p_i - k}{2} + 1\right).$$

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Among these, the number of those that do not satisfy condition (3) is

$$n_2 = 2\prod_{i=1}^{n}\left(\frac{p_i-k}{2}\right).$$

Likewise, the number of those that do not satisfy condition (4) is also n_2. Further, the number of those that satisfy neither (3) nor (4) is

$$n_3 = \prod_{i=1}^{t}\left(\frac{p_i-k}{2}-1\right)\prod_{i=t+1}^{n}\left(\frac{p_i-k}{2}\right) + \prod_{i=1}^{t}\left(\frac{p_i-k}{2}\right)\prod_{i=t+1}^{n}\left(\frac{p_i-k}{2}-1\right).$$

Hence, the number of n-tuples $(k_1,\ldots,k_n)$ satisfying conditions (1)–(4) is

$$N(k) = n_1 - 2n_2 + n_3.$$

On writing

$$q_i = \tfrac{1}{2}(p_i - k) \quad i = 1,\ldots,n$$

we have

$$\begin{aligned}
N(k) &= \prod_{i=1}^{t}(q_i+1)\prod_{i=t+1}^{n} q_i + \prod_{i=1}^{t} q_i \prod_{i=t+1}^{n}(q_i+1) - 2\prod_{i=1}^{n} q_i \\
&\quad + \prod_{i=1}^{t}(q_i-1)\prod_{i=t+1}^{n} q_i + \prod_{i=1}^{t} q_i \prod_{i=t+1}^{n}(q_i-1) \\
&= \left\{\prod_{i=1}^{t}(q_i+1) + \prod_{i=1}^{t}(q_i-1) - 2\prod_{i=1}^{t} q_i\right\}\prod_{i=t+1}^{n} q_i \\
&\quad + \prod_{i=1}^{t} q_i\left\{\prod_{i=t+1}^{n}(q_i+1) + \prod_{i=t+1}^{n}(q_i-1) - 2\prod_{i=t+1}^{n} q_i\right\} \\
&= 2\left\{\sum_{r=1}^{[t/2]} S_{t-2r}(q_1,\ldots,q_t)\,q_{t+1}\cdots q_n \right.\\
&\quad \left. + q_1\cdots q_t \sum_{r=1}^{[(n-t)/2]} S_{n-t-2r}(q_{t+1},\ldots,q_n)\right\}.
\end{aligned}$$

This is the number of matching blocks of width k for odd $k < p$.

(c) Finally, suppose k is even. Then the n-tuples $(k_1,\ldots,k_n)$ satisfying conditions (1) and (2) are given by

$$k_i = \begin{cases} 1,3,\ldots,p_i - k & (i = 1,\ldots,t), \\ 2,4,\ldots,p_i - k + 1 & (i = t+1,\ldots,n) \end{cases}$$

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or

$$k_i = \begin{cases} 2,4,\ldots,p_i-k+1 & (i=1,\ldots,t), \\ 1,3,\ldots,p_i-k & (i=t+1,\ldots,n). \end{cases}$$

The number of such n-tuples is

$$n_1 = 2\prod_{i=1}^{n}\left(\frac{p_i-k+1}{2}\right).$$

Among these, the number of those that do not satisfy (3), as well as of those that do not satisfy (4), is

$$n_2 = \prod_{i=1}^{t}\left(\frac{p_i-k+1}{2}\right)\prod_{i=t+1}^{n}\left(\frac{p_i-k-1}{2}\right) + \prod_{i=1}^{t}\left(\frac{p_i-k-1}{2}\right)\prod_{i=t+1}^{n}\left(\frac{p_i-k+1}{2}\right).$$

The number of those that satisfy neither (3) nor (4) is

$$n_3 = 2\prod_{i=1}^{n}\left(\frac{p_i-k-1}{2}\right).$$

Hence, on writing

$$q_i' = \frac{p_i-k-1}{2}$$

the number of n-tuples $(k_1,\ldots,k_n)$ satisfying condition (1)–(4) is

$$\begin{aligned} N(k) &= n_1 - 2n_2 + n_3 \\ &= 2\left\{\prod_{i=1}^{n}(q_i'+1) + \prod_{i=1}^{n} q_i'\right\} \\ &\quad - 2\left\{\prod_{i=1}^{t}(q_i'+1)\prod_{i=t+1}^{n} q_i' + \prod_{i=1}^{t} q_i'\prod_{i=t+1}^{n}(q_i'+1)\right\} \\ &= 2\left\{\prod_{i=1}^{n}(q_i'+1) - \prod_{i=1}^{n} q_i'\right\} - 2\left\{\prod_{i=1}^{t}(q_i'+1)\prod_{i=t+1}^{n} q_i' - \prod_{i=1}^{n} q_i'\right\} \\ &\quad - 2\left\{\prod_{i=1}^{t} q_i'\prod_{i=t+1}^{n}(q_i'+1) - \prod_{i=1}^{n} q_i'\right\} \end{aligned}$$

$$= 2\sum_{r=1}^{n} S_{n-r}(q'_1,\ldots,q'_n) - 2\left\{q'_1\cdots q'_t \sum_{r=1}^{n-t} S_{n-t-r}(q'_{t+1},\ldots,q'_n)\right.$$

$$\left. - q'_{t+1}\cdots q'_n \sum_{r=1}^{t} S_{t-r}(q'_1,\ldots,q'_t)\right\}.$$

This is the number of matching blocks of width k for even $k < p$. □

In the particular case $n = 2$, $t = 1$, we have

$$M = \tfrac{1}{2}(p_1 p_2 - 1),$$

$$N = \tfrac{1}{2}(p_1 + p_2 - 2),$$

$$N(p) = \tfrac{1}{2}|p_1 - p_2|,$$

$$N(k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

If $n = 3$, $t = 2$, we have

$$M = \tfrac{1}{4}(p_1 p_2 p_3 - p_1 - p_2 + p_3),$$

$$N = \tfrac{1}{4}(p_1 p_2 + p_1 p_3 + p_2 p_3 - 2p_1 - 2p_2 + 1),$$

$$N(p) = \begin{cases} \tfrac{1}{4}(|p_1 - p_2| + 2)(p_3 - p) & \text{if } p = p_1 \text{ or } p_2, \\ \tfrac{1}{4}(p_1 - p_3)(p_2 - p_3) & \text{if } p = p_3, \end{cases}$$

$$N(k) = \begin{cases} \tfrac{1}{2}(p_1 + p_2 - 2k)(p_3 - k) & \text{if } k \text{ is odd,} \\ p_1 + p_2 - 2k & \text{if } k \text{ is even.} \end{cases}$$

We now consider the case where $p_1,\ldots,p_n$ are coprimes but not all odd. Then exactly one of them is even and, without loss of generality, we may assume p_1 to be even.

Theorem 2. *Let $p_1,\ldots,p_n$ be coprimes, and suppose p_1 is even. Let $P = p_1\cdots p_n$, and $p = \min(p_1,\ldots,p_n)$. Let A be an $n\times P$ matrix in which the ith row has alternately p_i 0's and p_i 1's, and suppose that the first t rows start with 0's, and the remaining $n-t$ rows start with 1's. Then*

(a) *The number M of matching columns in A is*

$$M = \frac{1}{2^{n-1}}\left\{p_1\cdots p_n + (-1)^{n-t} p_1 \sum_{r=1}^{[(n-1)/2]} S_{n-2r-1}(p'_2,\ldots,p'_n)\right\},$$

where $p'_i = p_i$ for $i = 1,\ldots,t$, and $p'_i = -p_i$ for $i = t+1,\ldots,n$.

(b) *The number N of matching blocks in A is*

$$N = \frac{1}{2^{n-1}}\left\{(-1)^{n-1} p_1 \sum_{r=1}^{[(n-1)/2]} S_{n-2r-1}(p'_2,\ldots,p'_n) + \sum_{r=1}^{n} (-1)^r S_{n-r}(p_1,\ldots,p_n)\right\}.$$

(c) *The number of matching blocks of width p is*

$$N(p)=\begin{cases} \dfrac{1}{2^{n-1}}\displaystyle\prod_{i=1}^{n}(p_i-p+1) & \text{if } p=p_1,\\ \dfrac{(p_1-p+1)}{2^n}\displaystyle\prod_{i=2}^{t}(p_i-p+2)\prod_{i=t+1}^{n}(p_i-p) & \text{if } p\in\{p_2,\ldots,p_t\},\\ \dfrac{(p_1-p-1)}{2^n}\displaystyle\prod_{i=2}^{t}(p_i-p)\prod_{i=t+1}^{n}(p_i-p+2) & \text{if } p\in\{p_{t+1},\ldots,p_n\}. \end{cases}$$

(d) *For every odd positive integer $k < p$, the number of matching blocks of width k is*

$$\begin{aligned} N(k) = (p_1-k)\Bigg\{ & q_2\cdots q_t \sum_{r=1}^{[(n-t)/2]} S_{n-t-2r}(q_{t+1},\ldots,q_n) \\ & + q_{t+1}\cdots q_n \sum_{r=1}^{[(t-1)/2]} S_{t-2r-1}(q_2,\ldots,q_t)\Bigg\} \\ & + q_2\cdots q_t \sum_{r=0}^{[(n-t-1)/2]} S_{n-t-2r-1}(q_{t+1},\ldots,q_n) \\ & + q_{t+1}\cdots q_n \sum_{r=1}^{[t/2]} S_{t-2r}(q_2,\ldots,q_t), \end{aligned}$$

where

$$q_i = \tfrac{1}{2}(p_i - k) \quad i=1,\ldots,n.$$

(e) *For every even positive integer $k < p$*

$$N(k) = (p_1-k+1)\sum_{r=1}^{n-1} S_{n-r-1}(q'_2,\ldots,q'_n)$$

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$$-(p_1-k)\left\{q_2'\cdots q_t'\sum_{r=1}^{n-t} S_{n-t-r}(q_{t+1}',\dots,q_n')\right.$$

$$\left.-q_{t+1}'\cdots q_n'\sum_{r=1}^{t-1} S_{t-r-1}(q_2',\dots,q_t')\right\}.$$

where

$$q_i' = \frac{p_i - k - 1}{2} \quad i = 1,\dots,n.$$

Proof. The general technique for proving this theorem is the same as in the foregoing theorem. Therefore, we indicate only the significant points where the proof differs from the previous proof.

We consider an $n\times 2P$ matrix A', in which the entries have the same pattern as in A. Then, as already mentioned at the beginning of this section, the matrix A' has twice as many matching columns as A.

Given any n-tuple $(k_1,\dots,k_n)$ with $1\leqslant k_i\leqslant p_i$, $i=1,\dots,n$, let x be the unique integer (lying between 1 and $p_1\cdots p_n$) such that $x\equiv k_i \pmod{p_i}$ for each $i=1,\dots,n$. Then $x'=x+P$ is the unique integer lying between $P+1$ and $2P$, such that $x'\equiv k_i \pmod{p_i}$ for each $i=1,\dots,n$. Write $x=b_ip_i+k_i$. So $x'=b_i'p_i+k_i$, where

$$b_i' = b_i + \frac{P}{p_i} \quad (i=1,\dots,n).$$

Now P/p_1 is odd, hence b_1 and b_1' are of opposite parity, that is one of them is even and the other is odd. But for all $i=2,\dots,n$, the integers b_i and b_i' have the same parity. Therefore, one (and only one) of the columns x,x' is a matching column in A' if the following conditions hold:

(1) $1\leqslant k_i\leqslant p_i$, $i=1,\dots,n$,
(2) either $k_2,\dots,k_t$ are all odd and $k_{t+1},\dots,k_n$ are all even or $k_2,\dots,k_t$ are all even and $k_{t+1},\dots,k_n$ are all odd.

The number of matching columns in A' is equal to the number of n-tuples $(k_1,\dots,k_n)$ satisfying these conditions. The various results stated in the theorem can now be proved by using the counting procedures employed in the proof of the foregoing theorem. $\square$

In the particular case $n=2$, $t=1$, we have

$$M = \tfrac{1}{2}p_1p_2,$$

$$N = \tfrac{1}{2}(p_1+p_2-1),$$

$$N(p) = \tfrac{1}{2}(|p_1-p_2|+1),$$

$$N(k) = 1 \quad \text{for } k<p.$$

If $n=3$, $t=2$, then

$$M=\tfrac{1}{4}(p_1p_2p_3-p_1),$$

$$N=\tfrac{1}{4}(p_1p_2+p_1p_3+p_2p_3-2p_1-p_2-p_3+1),$$

$$N(p)=\begin{cases}\tfrac{1}{4}(p_2-p_1-1)(p_3-p_1+1) & \text{if } p=p_1,\\ \tfrac{1}{4}(p_1-p+1)|p_3-p_2| & \text{if } p=p_2,p_3,\end{cases}$$

$$N(k)=\begin{cases}\tfrac{1}{2}(p_2+p_3)-k & \text{if } k \text{ is odd},\\ p_1+\tfrac{1}{2}(p_2+p_3)-2k & \text{if } k \text{ is even}.\end{cases}$$

We now state the results for the special case where all rows start with the same color. The next two theorems follow from Theorems 1 and 2 on putting $t=n$.

Theorem 3. *Let $p_1,\ldots,p_n$ be odd coprimes, $P=p_1\cdots p_n$, and $p=\min(p_1,\ldots,p_n)$. Let A be an $n\times P$ matrix in which the ith row has alternately p_i 0's and p_i 1's, each row starting with 0's. Then the width of every matching block in A is an odd integer. Moreover,*

(a) *The number M of matching columns in A is*

$$M=\frac{1}{2^{n-1}}\left\{p_1\cdots p_n+\sum_{r=1}^{\lfloor n/2\rfloor}S_{n-2r}(p_1,\ldots,p_n)\right\}.$$

(b) *The total number of matching blocks in A is*

$$N=\frac{1}{2^{n-1}}\sum_{r=0}^{\lfloor (n-1)/2\rfloor}S_{n-2r-1}(p_1,\ldots,p_n).$$

(c) *The number of matching blocks of width p is*

$$N(p)=\frac{1}{2^n}\prod_{i=1}^{n}(p_i-p+2).$$

(d) *For every odd positive integer $k<p$, the number of matching blocks of width k is*

$$N(k)=2\sum_{r=1}^{\lfloor n/2\rfloor}S_{n-2r}(q_1,\ldots,q_n),$$

where

$$q_i=\tfrac{1}{2}(p_i-k)\quad \textit{for each } i=1,\ldots,n.$$

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In the particular case $n = 2$, we have

$$M = \tfrac{1}{2}(p_1 p_2 + 1),$$

$$N = \tfrac{1}{2}(p_1 + p_2),$$

$$N(p) = \tfrac{1}{2}(|p_1 - p_2| + 2),$$

$$N(k) = 2 \quad \text{for every odd } k < p.$$

For $n = 3$

$$M = \tfrac{1}{4}(p_1 p_2 p_3 + p_1 + p_2 + p_3),$$

$$N = \tfrac{1}{4}(p_1 p_2 + p_1 p_3 + p_2 p_3 + 1),$$

$$N(p) = \tfrac{1}{8}(p_1 - p + 2)(p_2 - p + 2)(p_3 - p + 2),$$

$$N(k) = p_1 + p_2 + p_3 - 3k \quad \text{for every odd } k < p.$$

Theorem 4. *Let $p_1, \ldots, p_n$ be coprimes, and suppose p_1 is even. Let $P = p_1 \cdots p_n$, and $p = \min(p_1, \ldots, p_n)$. Let A be an $n \times P$ matrix in which the ith row has alternately p_i 0's and p_i 1's, each row starting with 0's. Then*

(a) *The number M of matching columns in A is*

$$M = \frac{1}{2^{n-1}} \left\{ p_1 \cdots p_n + p_1 \sum_{r=1}^{[(n-1)/2]} S_{n-2r-1}(p_2, \ldots, p_n) \right\}.$$

(b) *The total number of matching blocks in A is*

$$N = \frac{1}{2^{n-1}} \left\{ p_1 \sum_{r=1}^{[(n-1)/2]} S_{n-2r-1}(p_2, \ldots, p_n) - \sum_{r=1}^{n} (-1)^r S_{n-r}(p_1, \ldots, p_n) \right\}.$$

(c) *The number of matching blocks of width p in A is*

$$N(p) = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\prod_{i=2}^{n} (p_i - p + 1) & \text{if } p = p_1, \\ \dfrac{1}{2^{n}} (p_1 - p + 1) \displaystyle\prod_{i=2}^{n} (p_i - p + 2) & \text{if } p \neq p_1. \end{cases}$$

(d) *For every odd positive integer $k < p$*

$$N(k) = (p_1 - k) \sum_{r=1}^{[(n-1)/2]} S_{n-1-2r}(q_2, \ldots, q_n) + \sum_{r=1}^{[n/2]} S_{n-2r}(q_2, \ldots, q_n),$$

where

$$q_i = \tfrac{1}{2}(p_i - k) \quad \textit{for each } i = 1, \ldots, n.$$

(e) *For every even positive integer $k < p$,*

$$N(k) = \sum_{r=1}^{n-1} S_{n-1-r}(q'_2, \ldots, q'_n),$$

where

$$q'_i = \tfrac{1}{2}(p_i - k - 1) \quad \textit{for each } i = 1, \ldots, n.$$

In the particular case $n = 2$, we have

$$M = \tfrac{1}{2} p_1 p_2,$$

$$N = \tfrac{1}{2}(p_1 + p_2 - 1),$$

$$N(p) = \tfrac{1}{2}(|p_1 - p_2| - 1),$$

$$N(k) = 1 \quad \text{for every } k < p.$$

For $n = 3$,

$$M = \tfrac{1}{4}(p_1 p_2 p_3 + p_1),$$

$$N = \tfrac{1}{4}(p_1 p_2 + p_1 p_3 + p_2 p_3 - p_2 - p_3 + 1),$$

$$N(p) = \begin{cases} \tfrac{1}{4}(p_2 - p + 1)(p_3 - p + 1) & \text{if } p = p_1, \\ \tfrac{1}{8}(p_1 - p - 1)(p_2 - p + 2)(p_3 - p + 2) & \text{if } p \neq p_1. \end{cases}$$

$$N(k) = \begin{cases} p_1 + \tfrac{1}{2}(p_2 + p_3) - 2k & \text{for every odd } k < p, \\ \tfrac{1}{2}(p_2 + p_3) - k & \text{for every even } k < p. \end{cases}$$

The results of the foregoing theorems can be generalized to the case where $p_1, \ldots, p_n$ are not coprimes, provided that the numbers obtained on dividing these by their greatest common divisor are coprimes. (If $n = 2$, this condition is satisfied automatically.)

Theorem 5. *Let $p_1, \ldots, p_n$ be positive integers and let $d = \gcd(p_1, \ldots, p_n)$. Let $p'_i = p_i/d$, $i = 1, \ldots, n$, and suppose that $p'_1, \ldots, p'_n$ are coprimes. Let A be an $n \times P$ matrix (where $P = p_1 \cdots p_n$) in which the ith row has alternately p_i 0's and p_i 1's. Let A' be an $n \times P'$ matrix (where $P' = p'_1 \cdots p'_n$) in which the ith row has alternately p'_i 0's and p'_i 1's. Further suppose that the ith rows in A and A' start with the same digit. Let M, M' be the numbers of matching columns in A, A', respectively. Then*

$$M = d^n M'$$

Likewise, if $N(k), N'(k)$ are the numbers of matching blocks of width k in A, A', respectively, then

$$N(dk) = d^{n-1} N'(k).$$

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Proof. Let us partition the matrix A into d^{n-1} submatrices $A_1, \ldots, A_{d^{n-1}}$, of the same size $n \times P'd$,

$$A = [A_1 \quad \cdots \quad A_{d^{n-1}}].$$

It is clear that each of these submatrices has the same number of matching columns. Now A_i can be obtained from A' by replacing each column in A' with d identical columns. Hence, the number of matching columns in A_i is dM'. Hence, the number of matching columns in A is $d^{n-1}dM' = d^nM'$. The second result is proved similarly. □

In conjunction with the previous theorems, this theorem provides the formula for M in various cases. For example, if $p'_1, \ldots, p'_n$ are odd coprimes, and each row in A begins with 0's, then on using Theorem 3, we have

$$M = \frac{d^n}{2^{n-1}}\left\{p'_1 \cdots p'_n + \sum_{r=1}^{[n/2]} S_{n-2r}(p'_1, \ldots, p'_n)\right\}.$$

$$= \frac{1}{2^{n-1}}\left\{p_1 \cdots p_n + \sum_{r=1}^{[n/2]} d^{2r} S_{n-2r}(p_1, \ldots, p_n)\right\}.$$

In particular, if $n = 2$, then

$$M = \tfrac{1}{2}(p_1 p_2 + d^2).$$

3. Matching columns and blocks with m colors

We now consider the general case of m colors, and assume that $p_1, \ldots, p_n$ are all coprimes and congruent to one another modulo m. (This corresponds to the restriction in the special case $m = 2$ that they are all odd.) We further assume that all rows start with the same color.

For any integer b, let $b \bmod m$ denote the remainder left on dividing b by m.

Theorem 6. *Let $m > 1$ and let $p_1, \ldots, p_n$ be coprimes, and suppose that $p_i \equiv s \pmod m$ for each $i = 1, \ldots, n$ (where $1 \leqslant s \leqslant m-1$). Let $P = p_1 \cdots p_n$ and $p = \min(p_1, \ldots, p_n)$. Let A be an $n \times P$ matrix in which the ith row has successively p_i 0's, p_i 1's, ..., p_i $(m-1)$'s, repeatedly, each row starting with 0's. Then*

(a) *The number M of matching columns in A is*

$$M = \frac{1}{m^{n-1}}\left\{p_1 \cdots p_n + \frac{s(m-s)}{m}\sum_{r=2}^{n}[(m-s)^{r-1} - (-s)^{r-1}]S_{n-r}(p_1, \ldots, p_n)\right\}.$$

(b) *The total number N of matching blocks in A is*

$$N = \frac{1}{m^n}\sum_{r=1}^{n}[(m-s)^r - (-s)^r]S_{n-r}(p_1,\ldots,p_n).$$

(c) *The number of matching blocks of width p is*

$$N(p) = \frac{1}{m^n}\prod_{i=1}^{n}(p_i - p + m).$$

(d) *The width k of every matching block is an integer congruent to $s \bmod m$, and for every such $k < p$, the number of matching blocks of width k is*

$$N(k) = 2\sum_{r=1}^{[n/2]} S_{n-2r}(q_1,\ldots,q_n),$$

where

$$q_i = \frac{p_i - k}{m} \quad i = 1,\ldots,n.$$

Proof. As before, given any integer x, $1 \leqslant x \leqslant P$, let $b_i p_i$ be the largest multiple of $p_i \leqslant x$. Then the entries in the matrix A are given by

$$a_{ix} = b_i \bmod m.$$

Suppose the xth column in A is a matching column. Then $b_1,\ldots,b_n$ are all congruent modulo m. Write $k_i = x - b_i p_i$. Because $p_1,\ldots,p_n$ are all congruent modulo m, it follows that $k_1,\ldots,k_n$ are all congruent modulo m. Conversely, if $k_1,\ldots,k_n$ are all congruent modulo m, then $b_1 p_1,\ldots,b_n p_n$ are all congruent modulo m. Since $p_1,\ldots,p_n$ are co-primes, s is relatively prime to m. Hence $b_1,\ldots,b_n$ are all congruent modulo m. Therefore, the number of matching columns is equal to the number of n-tuples $(k_1,\ldots,k_n)$ such that

(1) $1 \leqslant k_i \leqslant p_i$ $(i = 1,\ldots,n)$.
(2) $k_1,\ldots,k_n$ are all congruent modulo m.

Suppose $k_i \equiv r \pmod m$ for each $i = 1,\ldots,n$ (where $1 \leqslant r \leqslant m$). If $1 \leqslant r \leqslant s$, then

$$k_i = r, r+m,\ldots,p_i - s + r.$$

But if $s+1 \leqslant r \leqslant m$, then

$$k_i = r, r+m,\ldots,p_i - m - s + r.$$

The various results stated in the theorem can now be proved by following the counting procedures used in the proof of Theorem 1. $\square$

In the particular case $n=2$, we have

$$M = \frac{1}{m}\{p_1p_2 + s(m-s)\},$$

$$N = \frac{1}{m}(p_1 + p_2 + m - 2s),$$

$$N(p) = \frac{1}{m}(|p_1 - p_2| + m),$$

$$N(k) = 2 \quad \text{for } k<p,\ k \equiv s \pmod m.$$

If $n=3$, then

$$M = \frac{1}{m^2}\{p_1p_2p_3 + s(m-s)(p_1+p_2+p_3) + m - 2s\},$$

$$N = \frac{1}{m^2}\{p_1p_2 + p_1p_3 + p_2p_3 + (m-2s)(p_1+p_2+p_3) + m^2 - 3ms + 3s^2\},$$

$$N(p) = \frac{1}{m^2}(p_1 - p + m)(p_2 - p + m)(p_3 - p + m),$$

$$N(k) = \frac{2}{m}(p_1 + p_2 + p_3 - 3k) \quad \text{for } k<p,\ k \equiv s \pmod m.$$

We now consider the special case $n=2$ and m colors, and drop the condition that $p_1 \equiv p_2 \pmod m$. Instead, we assume that $p_1 - p_2$ is relatively prime to m. Further, to ensure symmetry with respect to the m colors, we must in this case consider a matrix with mp_1p_2 columns.

Theorem 7. *Let p_1, p_2 be coprimes and suppose that $p_1 - p_2$ is relatively prime to m. Let A be a $2 \times mp_1p_2$ matrix in which the ith row has repeated successions of p_i 0's, p_i 1's, ..., p_i $(m-1)$'s, $i=1,2$, and each row starts with 0's. Then*

(a) *The number M of matching columns in A is*

$$M = p_1p_2.$$

(b) *The total number of matching blocks in A is*

$$N = p_1 + p_2 - 1.$$

(c) *The number of matching blocks of width* $p = \min(p_1, p_2)$ is

$$N(p) = |p_1 - p_2| + 1.$$

(d) *The number of matching blocks of width $k<p$ is*

$$N(k) = 2.$$

Proof. (a) Given any x $(1 \leqslant x \leqslant mp_1p_2)$, let b_ip_i be the largest multiple of $p_i < x$ $(i=1,2)$. Then the column x in A is a matching column if and only if $b_1 \equiv b_2 \pmod m$.

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Consider any ordered pair (k_1, k_2), where $1 \leqslant k_1 \leqslant p_1$ and $1 \leqslant k_2 \leqslant p_2$. By Chinese remainder theorem, there is a unique number x_0 (lying between 1 and $p_1 p_2$) such that $x_0 \equiv k_i \pmod{p_i}$, $i = 1, 2$. So there exist b_1, b_2 such that $x_0 = k_1 + b_1 p_1 = k_2 + b_2 p_2$. Let

$$x_r = x_0 + r p_1 p_2 \quad r = 0, 1, \ldots, m-1.$$

Then $x_0, \ldots, x_{m-1}$ are the only solutions, lying between 1 and $m p_1 p_2$, of the congruences $x \equiv k_i \pmod{p_i}$, $i = 1, 2$. Now

$$x_r = k_1 + (b_1 + r p_2) p_1 = k_2 + (b_2 + r p_1) p_2.$$

So the column x_r is a matching column if and only if $b_1 + r p_2 \equiv b_2 + r p_1 \pmod{m}$.

Since, by hypothesis, $p_1 - p_2$ is relatively prime to m, there exists s such that

$$s(p_1 - p_2) \equiv 1 \pmod{m}.$$

Let

$$r = ((b_1 - b_2)s) \bmod m.$$

Then

$$\begin{aligned} r(p_1 - p_2) &\equiv (b_1 - b_2)s(p_1 - p_2) \pmod{m} \\ &\equiv (b_1 - b_2) \pmod{m}. \end{aligned}$$

Hence

$$b_1 + r p_2 \equiv b_2 + r p_1 \pmod{m}$$

and therefore column x_r is a matching column. Conversely, if column x_r is a matching column, then it is easily seen that $r \equiv (b_1 - b_2)s \pmod{m}$.

Thus, we see that there is a unique matching column in A corresponding to every ordered pair (k_1, k_2) such that $1 \leqslant k_1 \leqslant p_1$ and $1 \leqslant k_2 \leqslant p_2$. Hence, the number of matching columns in A is equal to $p_1 p_2$.

The other parts of the theorem are proved similarly. □

From Theorems 6 and 7, we get the following corollary when $n = 2$ and m is prime.

Corollary 8. *Let p_1, p_2 be coprimes and m prime. Let A be a $2 \times m p_1 p_2$ matrix in which the ith row has repeated successions of p_i 0's, p_i 1's, …, p_i $(m-1)$'s, $i = 1, 2$, and each row starts with 0's. Let M be the number of matching columns in A. Then*

(a) *If $p_1 \equiv p_2 \equiv s \pmod{m}$, $1 \leqslant s \leqslant m - 1$, then*

$$M = p_1 p_2 + s(m - s).$$

(b) *If $p_1 \not\equiv p_2 \pmod{m}$, then*

$$M = p_1 p_2.$$

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4. Two identities in $p_1, \ldots, p_n$

In this section we obtain two identities in $p_1, \ldots, p_n$ as a byproduct of Theorem 6. On summing $N(k)$ and $kN(k)$ over all values of k, we have

$$N = N(p) + \sum_{k<p} N(k),$$

$$M = pN(p) + \sum_{k<p} kN(k),$$

which on using the formulas for $M, N, N(p), N(k)$ obtained in Theorem 6, yields two identities in $p_1, \ldots, p_n$. To obtain these identities, we first express $N(k)$ explicitly in terms of $p_1, \ldots, p_n$ as follows. On using the relations $q_i = (1/m)(p_i - k)$, $i = 1, \ldots, n$, we have

$$q_1 \cdots q_i = \frac{1}{m^i} \sum_{j=0}^{i} (-1)^j k^j S_{i-j}(p_1, \ldots, p_i).$$

Hence

$$S_i(q_1, \ldots, q_n) = \frac{1}{m^i} \sum_{j=0}^{i} (-1)^j k^j \binom{n-i+j}{j} S_{i-j}(p_1, \ldots, p_n).$$

Therefore

$$\begin{aligned} N(k) &= 2 \sum_{r=1}^{[n/2]} S_{n-2r}(q_1, \ldots, q_n) \\ &= \frac{2}{m^n} \sum_{r=1}^{[n/2]} m^{2r} \sum_{j=0}^{n-2r} (-1)^j k^j \binom{2r+j}{j} S_{n-2r-j}(p_1, \ldots, p_n). \end{aligned}$$

On changing the order of summation over r and j, we have

$$N(k) = \frac{2}{m^n} \sum_{j=0}^{n-2} (-1)^j A_j k^j,$$

where

$$A_j = \sum_{r=1}^{[(n-j)/2]} m^{2r} \binom{2r+j}{j} S_{n-2r-j}(p_1, \ldots, p_n).$$

On summing up $N(k)$ over $k=s,\ s+m,\dots,p-m$, we get

$$\sum_{k=s}^{p-m} N(k) = \frac{2}{m^n}\sum_{j=0}^{n-2}(-1)^j A_j B_j(p),$$

where

$$B_j(p) = \sum_{k<p,\,k\equiv s\,(\mathrm{mod}\,m)} k^j = s^j + (s+m)^j + \cdots + (p-m)^j.$$

Thus the number of matching blocks in A is

$$N = \frac{1}{m^n}\prod_{i=1}^{n}(p_i - p + m) + \frac{2}{m^n}\sum_{j=0}^{n-2}(-1)^j A_j B_j(p).$$

Now $p = \min(p_1,\dots,p_n)$. But the expression for N obtained in Theorem 6 is symmetric in $p_1,\dots,p_n$. Hence p can be taken to be any one of $p_1,\dots,p_n$. On taking $p=p_1$, we have

$$N = \frac{1}{m^n}\prod_{i=1}^{n}(p_i - p_1 - m) + \frac{2}{m^n}\sum_{j=0}^{n-2}(-1)^j A_j B_j(p_1).$$

On equating this with the value for N obtained in the theorem, we get the following identity:

$$\prod_{i=1}^{n}(p_i - p_1 - m) + 2\sum_{j=0}^{n-2}(-1)^j A_j B_j(p_1) = \sum_{r=1}^{n}\lambda_r S_{n-r}(p_1,\dots,p_n), \tag{1}$$

where

$$\lambda_r = (m-s)^r - (-s)^r.$$

Likewise, on equating the two expressions for M, we get the following identity:

$$p_1\prod_{i=1}^{n}(p_i - p_1 + m) + 2\sum_{j=0}^{n-2}(-1)^j A_j B_{j+1}(p_1)$$

$$= m p_1 \cdots p_n + s(m-s)\sum_{r=2}^{n}\lambda_{r-1} S_{n-r}(p_1,\dots,p_n). \tag{2}$$

These are identities between polynomials in $p_1,\dots,p_n$. Hence they hold irrespective of the restrictions on $p_1,\dots,p_n$ under which Theorem 6 was proved. On putting $p_1=p$ and $p_i=0$ for $i=2,\dots,n$ in (1) and (2), we get the following result.

Theorem 9. *Let p be a positive integer, $p \equiv s \pmod m$, $1 \leqslant s \leqslant m-1$. For any positive integer j, let $B_j(p)$ denote the sum of jth powers of positive integers less than p and congruent to s mod m, that is*

$$B_j(p) = \sum_{k<p,\, k\equiv s \pmod m} k^j = s^j + (s-m)^j + \cdots + (p-m)^j.$$

Then for any integer $n > 1$

(a) $$2\sum_{j=0}^{n-2}(-1)^j a_j B_j(p) = \lambda_n + \lambda_{n-1}p - m(m-p)^{n-1}.$$

(b) $$2\sum_{j=0}^{n-2}(-1)^j a_j B_{j+1}(p) = s(m-s)(\lambda_{n-1}+\lambda_{n-2}p) - pm(m-p)^{n-1}.$$

where

$$a_j = \begin{cases} m^{n-j}\binom{n}{j} & \text{if } n-j \text{ is even,} \\ m^{n-j-1}\binom{n-1}{j}p & \text{if } n-j \text{ is odd} \end{cases}$$

and

$$\lambda_r = (m-s)^r - (-s)^r.$$

We conclude with the remark that it would be interesting to solve the general case of m colors when the coprimes $p_1, \ldots, p_n$ are not all congruent to one another modulo m, and the rows do not necessarily start with the same color.

References

[1] T. Jiang, P. Kearney, M. Li, Open problems in computational molecular biology, Sigact News 30(3) (1999) 43–49.