

AN EXAMPLE OF A RIGHT  $q$ -RING

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## ABSTRACT

We show that Ivanov's classification of indecomposable non-local right  $q$ -rings is incomplete and provide a complete classification. Next, we correct and sharpen Byrd's classification of right  $q$ -rings.

**1. Introduction**

Given a ring  $R$ , we denote by  $\text{Soc}(R)$  and  $J(R)$  the right socle and the Jacobson radical of  $R$  respectively. Given a nonempty subset  $S$  of  $R$ , we set

$$r(R; S) = \{\alpha \in R \mid S\alpha = 0\} \quad \text{and} \quad \ell(R; S) = \{\alpha \in R \mid \alpha S = 0\}.$$

Recall that a ring  $R$  is said to be a right  $q$ -ring provided that all its right ideals are quasi-injective right modules. The study of right  $q$ -rings was initiated by Jain, Mohamed and Singh [9] in 1969. In particular they proved the following result which we shall need in the sequel.

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THEOREM 1.1 ([9, Theorem 2.3]): *Let  $R$  be a ring. Then  $R$  is a right  $q$ -ring if and only if  $R$  is a right self-injective ring and every essential right ideal of  $R$  is an ideal.*

Since then right  $q$ -rings have been studied in a number of papers [3, 4, 5, 6, 8, 10, 11, 12, 13]. The reader is referred to [7] for a survey on the subject.

Let  $n$  be an integer with  $n > 1$ , let  $D, D_1, D_2, \dots, D_n$  be skew fields and let  $V_{ij}$  be a  $D_i$ - $D_j$ -bimodule such that  $\dim(D, V_{ij}) = 1 = \dim(\{V_{ij}\}_D)$ ,  $1 \leq i, j \leq n$ . Denote by  $M_n(D_1, \dots, D_n; V_{12}, \dots, V_{n-1,n}, V_{n1})$  the ring of  $n \times n$  matrices of the form

$$\begin{pmatrix} D_1 & V_{12} & & & & \\ & D_2 & V_{23} & & & \\ & & D_3 & V_{34} & & \\ & & & \ddots & \ddots & \\ & & & & D_{n-1} & V_{n-1,n} \\ V_{n1} & & & & & D_n \end{pmatrix}$$

where it is understood that  $V_{ij}V_{pq} = 0$  for all  $i, j, p, q$ . Next, let  $\alpha$  be automorphisms of the skew field  $D$ . We denote by  $V$  the  $D$ - $D$ -bimodule  $D$  and by  $V(\alpha)$  the  $D$ - $D$ -bimodule which as a left  $D$ -module is equal to  ${}_D D$  and the right  $D$ -module structure is given by  $x * y = x\alpha(y)$  for all  $x \in V(\alpha)$ . We set

$$H(n; D; \alpha) = M_n(D, \dots, D; V, \dots, V, V(\alpha)).$$

In 1972 Ivanov [4, Theorem 3] proved that an indecomposable right  $q$ -ring either is simple Artinian, or is isomorphic to a ring  $H(n; D; \text{id}_D)$ . Recently Jain, López-Permouth and Syed [8, Theorem 3.13] obtained the following result.

THEOREM 1.2: *Let  $R$  be an indecomposable non-local right  $q$ -ring. Then  $R$  is of the form  $M_n(D_1, \dots, D_n; V_{12}, \dots, V_{n-1,n}, V_{n1})$  or simple Artinian. Conversely, every ring of the form  $M_n(D_1, \dots, D_n; V_{12}, \dots, V_{n-1,n}, V_{n1})$  or every simple Artinian ring is a right  $q$ -ring.*

We are now in a position to state our first main result.

THEOREM 1.3:

- (1) *Every ring  $M_n(D_1, \dots, D_n; V_{12}, \dots, V_{n-1,n}, V_{n1})$  is isomorphic to a ring of the form  $H(n; D; \alpha)$ .*
- (2) *Two rings  $H(n; D; \alpha)$  and  $H(n; D; \beta)$  are isomorphic if and only if there exists an automorphism  $\gamma$  of the skew field  $D$  such that  $\gamma^{-1}\beta\gamma\alpha^{-1}$  is an inner automorphism of  $D$ .*

*Example:* Let  $C$  be the complex number field and let  $\alpha$  be the complex conjugation. Clearly  $\alpha$  is not an inner automorphism of  $C$  and so  $H(2; C; \alpha) \not\cong H(2; C; \text{id})$  by Theorem 1.3.

The Example shows that the Ivanov's description of an indecomposable non-local right  $q$ -rings is not complete. It has to be noted that he made a mistake at the very end of the proof (he wrongly stated that any two  $D$ - $D$ -bimodules are isomorphic) and so his proof can be easily corrected.

Let  $n$  be a positive integer, let  $\Delta$  be a right  $q$ -ring with essential maximal right ideal  $P$  such that the module  $\Delta/P$  is injective, and clearly it cannot be embedded in  ${}_{\Delta}\Delta$ . Since  $P$  is an essential right ideal of  $\Delta$ , it is a two sided ideal by Theorem 1.1 and so  $D = \Delta/P$  is a skew field. Next, let  $V = {}_D D_D$  be a  $D$ - $D$ -bimodule. Clearly  $V$  is also a  $D$ - $\Delta$ -bimodule. We denote by  $G(n; \Delta; P)$  the ring of  $(n + 1) \times (n + 1)$  matrices of the form

$$\begin{pmatrix} D & V & & & & \\ & D & V & & & \\ & & D & V & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & D & V \\ & & & & & & \Delta \end{pmatrix}$$

where it is understood that  $VV = 0$ . Note that  $G(0; \Delta; P) = \Delta$ .

Ivanov [1] conjectured that every right  $q$ -ring is the direct sum of a finite number of indecomposable non-local right  $q$ -rings and a right  $q$ -ring all of whose idempotents are central. Byrd [1] classified right  $q$ -rings and showed that the structure of right  $q$ -rings is more complicated than it was conjectured by Ivanov. We state his main result [1, Theorem 6] in the following slightly different but equivalent form.

**THEOREM 1.4:** *A right  $q$ -ring is isomorphic to a finite direct product of right  $q$ -rings of the following kinds:*

- (1) *Semisimple Artinian ring.*
- (2)  *$H(n; D; \text{id}_D)$ , where  $\text{id}_D$  is the identity automorphism of  $D$ .*
- (3)  *$G(n; \Delta; P)$ , where  $\Delta$  is a right  $q$ -ring all of whose idempotents are central.*
- (4) *A right  $q$ -ring all of whose idempotents are central.*

In view of the Example, Byrd's classification of right  $q$ -rings is not complete (see Theorem 1.4(2)). It has to be noted that he made the same mistake as Ivanov in [1, Theorem 3] and his proof can be easily corrected. The main goal of the present paper is to correct and sharpen his classification.



by the rule

$$\beta([d_1, \dots, d_n, v_1, v_2, \dots, v_n]) = [d_1, \gamma(d_2), d_3, \dots, d_n, v_1v, v_2, \dots, v_n]$$

(note that  $v_1v$  is defined because  $v \in V = {}_D D_D$ ). Clearly  $\beta$  is a bijective additive map. Next, it follows from (1) that

$$\begin{aligned} &\beta([d_1, \dots, d_n, v_1, \dots, v_n][d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \\ &= \beta([d_1d'_1, \dots, d_nd'_n, d_1v'_1 + v_1d'_2, \gamma(d_2)v'_2 + v_2d'_3, \dots, v_nd'_1 - d_nv'_n]) \\ &= [d_1d'_1, \gamma(d_2d'_2), d_3d'_3, \dots, (d_1v'_1 + v_1d'_2)v, \gamma(d_2)v'_2 + v_2d'_3, \dots, v_nd'_1 + d_nv'_n] \\ &= [d_1d'_1, \gamma(d_2)\gamma(d'_2), d_3d'_3, \dots, d_1v'_1v + v_1v\gamma(d'_2), \gamma(d_2)v'_2 + v_2d'_3, \dots] \\ &= [d_1, \gamma(d_2), d_3, \dots, d_n, v_1v, v_2, \dots, v_n][d'_1, \gamma(d'_2), d'_3, \dots, d'_n, v'_1v, v'_2, \dots, v'_n] \\ &= \beta([d_1, \dots, d_n, v_1, \dots, v_n])\beta([d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \end{aligned}$$

and so  $\beta$  is an isomorphism of rings. Continuing in this fashion we get that  $R \cong M_n(D, \dots, D; V, \dots, V, V_{n1})$ .

Fix  $0 \neq w \in V_{n1}$ . Arguing as above we see that there exists an automorphism  $\alpha$  of the skew field  $D$  such that  $d w = w \alpha(d)$  for all  $d \in D$ . We now define a map  $\omega: H(n; D; \alpha^{-1}) \rightarrow M_n(D, \dots, D; V, \dots, V, V_{n1})$  by the rule

$$\omega([d_1, \dots, d_n, v_1, v_2, \dots, v_n]) = [d_1, d_2, d_3, \dots, d_n, v_1, \dots, v_{n-1}, v_n w].$$

Obviously  $\omega$  is a bijective additive map. Again making use of (1) we get

$$\begin{aligned} &\omega([d_1, \dots, d_n, v_1, \dots, v_n][d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \\ &= \omega([d_1d'_1, \dots, d_1v'_1 + v_1d'_2, \dots, d_{n-1}v'_{n-1} + v_{n-1}d'_n, v_n\alpha^{-1}(d'_1) + d_nv'_n]) \\ &= [d_1d'_1, \dots, d_1v'_1 + v_1d'_2, \dots, d_{n-1}v'_{n-1} + v_{n-1}d'_{n-1}, \{v_n\alpha^{-1}(d'_1) + d_nv'_n\}w] \\ &= [d_1d'_1, \dots, d_1v'_1 + v_1d'_2, \dots, d_{n-1}v'_{n-1} + v_{n-1}d'_{n-1}, v_n w d'_1 + d_n v'_n w] \\ &= [d_1, \dots, d_n, v_1, v_2, \dots, v_n w][d'_1, \dots, d'_n, v'_1, v'_2, \dots, v'_n w] \\ &= \omega([d_1, \dots, d_n, v_1, \dots, v_n])\omega([d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \end{aligned}$$

which shows that  $\omega$  is an isomorphism of rings. Thus  $R \simeq H(n; D; \alpha^{-1})$  and the first statement of Theorem 1.3 is proved.

We shall now prove the second statement of the theorem. We now assume that  $\alpha$  and  $\beta$  are isomorphisms of the skew field  $D$  such that  $H(n; D; \alpha) \simeq H(n; D; \beta)$ .

Let  $f: H(n; D; \alpha) \rightarrow H(n; D; \beta)$  be an isomorphism of rings.

Given a positive integer  $i$  with  $1 < i \leq n$ , we denote by  $e_i$  (respectively,  $e'_i$ ) the matrix in  $S = H(n; D; \alpha)$  (respectively,  $S' = H(n; D; \beta)$ ) whose  $(i, i)$  entry is 1 and all other entries zero. We set  $D_i = e_i S e_i$  and  $D'_i = e'_i S' e'_i$ ,  $i =$

$1, 2, \dots, n$ . Further, we put  $L_i = e_i S e_{i+1}$  and  $L'_i = e'_i S' e'_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ , and  $L_n = e_n S e_1$ ,  $L'_n = e'_n S' e'_1$ . Clearly both  $D_i$  and  $D'_i$  are skew fields isomorphic to  $D$  for all  $i = 1, 2, \dots, n$ . One can easily check that each  $L_i$  (respectively,  $L'_i$ ) is an ideal of  $S$  (respectively,  $S'$ ) which is simple as both left and right  $S$ -module (respectively,  $S'$ -module). We see that  $L_1, L_2, \dots, L_n$  (respectively,  $L'_1, L'_2, \dots, L'_n$ ) are all homogeneous components of the socle of  $S$  (respectively,  $S'$ ). Therefore there exists a permutation  $\sigma$  of the set  $\{1, 2, \dots, n\}$  such that  $f(L_i) = L'_{\sigma(i)}$  for all  $i = 1, 2, \dots, n$ .

Set  $L = \sum_{i=1}^n L_i$  and  $L' = \sum_{i=1}^n L'_i$  and note that  $\text{Soc}(S) = L$ ,  $\text{Soc}(S') = L'$  and

$$(2) \quad f(L) = L'.$$

Set  $T = \sum_{i=1}^n D_i \subseteq S$  and  $T' = \sum_{i=1}^n D'_i \subseteq S'$ . Clearly both  $T$  and  $T'$  are subrings of  $S$  and  $S'$  respectively. Moreover,  $T = \bigoplus_{i=1}^n D_i$  and  $T' = \bigoplus_{i=1}^n D'_i$ . Obviously  $S = T \oplus L$  and  $S' = T' \oplus L'$ . Let  $\pi: S \rightarrow T$  and  $\pi': S' \rightarrow T'$  be canonical projections of abelian groups. Clearly both  $\pi$  and  $\pi'$  are homomorphisms of ring. Since  $\ker(\pi) = L$  and  $\ker(\pi') = L'$ , (2) implies that  $f$  induces an isomorphism  $g: T \rightarrow T'$  of rings such that

$$(3) \quad g\pi = \pi'f.$$

Further, since both  $L^2 = 0$  and  $(L')^2 = 0$ ,

$$(4) \quad s\ell = \pi(s)\ell, \quad \ell s = \ell\pi(s), \quad s'\ell' = \pi'(s')\ell' \quad \text{and} \quad \ell's' = \ell'\pi'(s')$$

for all  $s \in S$ ,  $\ell \in L$ ,  $s' \in S'$  and  $\ell' \in L'$ . It now follows from both (3) and (4) that

$$f(t\ell) - f(t)f(\ell) = (\pi'f)(t)f(\ell) = (g\pi)(t)f(\ell) = g(t)f(\ell)$$

for all  $t \in T$  and  $\ell \in L$ . We see that

$$(5) \quad f(t\ell) = g(t)f(\ell) \quad \text{and} \quad f(\ell t) = f(\ell)g(t) \quad \text{for all } t \in T \text{ and } \ell \in L.$$

Since the ring  $T$  (respectively,  $T'$ ) is the direct sum of skew fields  $D_i$  (respectively,  $D'_i$ ),  $i = 1, 2, \dots, n$ , there exists a permutation  $\tau$  of the set  $\{1, 2, \dots, n\}$  such that  $g(D_i) = D'_{\tau(i)}$  for all  $i = 1, 2, \dots, n$ . We claim that  $\tau = \sigma$ . Indeed, assume that  $\tau(i) \neq \sigma(i)$  for some  $1 \leq i \leq n$ . Take  $0 \neq t \in D_i$  and  $0 \neq \ell \in L_i$ . Then  $t\ell \neq 0$  and so (5) implies that

$$0 \neq f(t\ell) = g(t)f(\ell) \in D'_{\tau(i)}L'_{\sigma(i)} = 0,$$

a contradiction. Therefore  $\tau = \sigma$  and so

$$(6) \quad g(D_i) = D'_{\sigma(i)}, \quad i = 1, 2, \dots, n.$$

Let  $1 \leq i \leq n$  and  $d \in D$ . We denote by  $d^{(i)}$  the  $2n$ -tuple  $[0, \dots, 0, d, 0, \dots, 0]$  where  $d$  is on  $i$ th position. Analogously, given  $1 \leq i \leq n - 1$  and  $v \in V$ , we denote by  $v^{(n+i)}$  the  $2n$ -tuple  $[0, \dots, 0, v, 0, \dots, 0]$  where  $v$  is on  $(n + i)$ th position. Finally, given  $v \in V(\alpha)$ ,  $v^{(2n)} = [0, \dots, 0, v]$ . Clearly

$$(7) \quad \begin{aligned} d^{(i)}v^{(n+i)} &= (dv)^{(n+i)} \quad \text{for all } i = 1, 2, \dots, n; \\ v^{(n+i)}d^{(i+1)} &= (vd)^{(n+i)} \quad \text{for all } i = 1, 2, \dots, n - 1; \\ v^{(2n)}d^{(1)} &= (v\alpha(d))^{(2n)}. \end{aligned}$$

Given  $1 \leq i \leq n$ , in view of (6),  $g$  induces an automorphism  $g_i$  of the skew field  $D$  such that

$$(8) \quad g(d^{(i)}) = \{g_i(d)\}^{(\sigma(i))} \quad \text{for all } d \in D.$$

Analogously,  $f$  induces an automorphism  $f_i$  of the additive group of  $D$  such that

$$(9) \quad f(v^{(n+i)}) = \{f_i(v)\}^{(n+\sigma(i))} \quad \text{for all } v \in D.$$

Given  $d, v \in D$ , we claim that

$$(10) \quad \begin{aligned} f_i(dv) &= g_i(d)f_i(v) \quad \text{for all } i = 1, 2, \dots, n; \\ f_i(vd) &= \begin{cases} f_i(v)g_{i+1}(d) & \text{if } i \neq n \text{ and } \sigma(i) \neq n; \\ f_n(v)(g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \neq n; \\ f_i(v)(\beta g_{i+1})(d) & \text{if } i \neq n \text{ and } \sigma(i) = n; \\ f_n(v)(\beta g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n. \end{cases} \end{aligned}$$

Indeed, it follows from (5), (7), (8) and (9) together that

$$\begin{aligned} \{f_i(dv)\}^{(n+\sigma(i))} &= f(\{dv\}^{(n+i)}) = f(d^{(i)}v^{(n+i)}) = g(d^{(i)})f(v^{(n+i)}) \\ &= \{g_i(d)\}^{(\sigma(i))}\{f_i(v)\}^{(n+\sigma(i))} = \{g_i(d)f_i(v)\}^{(n+\sigma(i))} \end{aligned}$$

which proves the first equality. The first case of the second equality in (10) is proved analogously. Now assume that  $i = n$  and  $\sigma(n) \neq n$ . We have

$$\begin{aligned} \{f_n(vd)\}^{(n+\sigma(n))} &= f(\{vd\}^{(2n)}) = f(v^{(2n)}\{\alpha^{-1}(d)\}^{(1)}) \\ &= f(v^{(2n)})g(\{\alpha^{-1}(d)\}^{(1)}) \\ &= \{f_n(v)\}^{(n+\sigma(n))}\{(g_1\alpha^{-1})(d)\}^{(\sigma(1))} \\ &= \{f_n(v)(g_1\alpha^{-1})(d)\}^{(n+\sigma(n))} \end{aligned}$$

(note that  $\sigma(n) < n$  forces  $\sigma(1) = \sigma(n) + 1$ ). The last two cases of the second equality in (10) are proved analogously.

Setting  $v = 1$  and  $t_i = g_i(1)$ ,  $i = 1, 2, \dots, n$ , we get from (10) that

$$f_i(d) = g_i(d)t_i \quad \text{for all } i = 1, 2, \dots, n;$$

$$f_i(d) = \begin{cases} t_i g_{i+1}(d) & \text{if } i \neq n \text{ and } \sigma(i) \neq n; \\ t_n(g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \neq n; \\ t_i(\beta g_{i+1})(d) & \text{if } i \neq n \text{ and } \sigma(i) = n; \\ t_n(\beta g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n \end{cases}$$

for all  $d \in D$  and so

$$(11) \quad g_i(d)t_i = \begin{cases} t_i g_{i+1}(d) & \text{if } i \neq n \text{ and } \sigma(i) \neq n; \\ t_n(g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \neq n; \\ t_i(\beta g_{i+1})(d) & \text{if } i \neq n \text{ and } \sigma(i) = n; \\ t_n(\beta g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n. \end{cases}$$

Let  $G$  be the group of all automorphisms of the skew field  $D$  and let  $N$  be the subgroup of all inner automorphisms of  $D$ . It is well-known that  $N$  is a normal subgroup of  $G$ . Let  $g \mapsto \bar{g}$  be the canonical projection of groups  $G \rightarrow G/N$ . Note that each  $g_i \in G$ .

There are two cases to consider.

CASE 1: Suppose that  $\sigma(j) = n$  for some  $1 \leq j < n$ . It follows from (11) that

$$\bar{g}_1 = \bar{g}_2 = \dots = \bar{g}_j = \overline{\beta g_{j+1}} \quad \text{and} \quad \bar{g}_{j+1} = \dots = \bar{g}_{n-1} = \bar{g}_n = \overline{g_1 \alpha^{-1}},$$

and so  $\overline{\beta g_{j+1}} - \bar{g}_1 = \overline{g_{j+1} \alpha}$  forcing  $\gamma^{-1} \beta \gamma \alpha^{-1} \in N$  with  $\gamma = g_{j+1}$ .

CASE 2: Suppose that  $\sigma(n) = n$ . According to (11) we have that

$$\bar{g}_1 = \bar{g}_2 = \dots = \bar{g}_{n-1} = \bar{g}_n = \overline{\beta g_1 \alpha^{-1}}$$

and so  $\bar{g}_1 = \overline{\beta g_1 \alpha^{-1}}$  forcing  $\gamma^{-1} \beta \gamma \alpha^{-1} \in N$  with  $\gamma = g_1$ .

To complete the proof of the theorem, assume that  $\gamma^{-1} \beta \gamma \alpha^{-1} \in N$  for some  $\gamma \in G$ . Set  $\delta = \gamma^{-1} \beta \gamma$ . Define a map  $f: H(n; D; \beta) \rightarrow H(n; D; \delta)$  by the rule

$$f([d_1, \dots, d_n, v_1, \dots, v_n]) = [\gamma^{-1}(d_1), \dots, \gamma^{-1}(d_n), \gamma^{-1}(v_1), \dots, \gamma^{-1}(v_n)].$$

Clearly  $f$  is a bijective additive map. It follows from (1) directly that  $f$  is an isomorphism of rings. Therefore it is enough to show that  $H(n; D; \delta) \cong H(n; D; \alpha)$ . Choose a nonzero element  $t \in D$  such that  $\delta(d) = t\alpha(d)t^{-1}$  for all  $d \in D$ . Clearly  $\delta(d)t = t\alpha(d)$ ,  $d \in D$ . Define a map  $g: H(n; D; \delta) \rightarrow H(n; D; \alpha)$  by the rule

$$g([d_1, \dots, d_n, v_1, \dots, v_n]) = [d_1, \dots, d_n, v_1, \dots, v_{n-1}, v_n t].$$



Obviously  $g$  is a bijective additive map. It now follows from (1) that

$$\begin{aligned} &g([d_1, \dots, d_n, v_1, \dots, v_n][d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \\ &= g([(d_1 d'_1, \dots, d_n d'_n, d_1 v'_1 + v_1 d'_2, \dots, d_{n-1} v'_{n-1} + v_{n-1} d'_n, v_n \delta(d'_1) + d_n v'_n)]) \\ &= [d_1 d'_1, \dots, d_n d'_n, d_1 v'_1 + v_1 d'_2, \dots, d_{n-1} v'_{n-1} + v_{n-1} d'_n, (v_n \delta(d'_1) + d_n v'_n) t'_n] \\ &= [d_1 d'_1, \dots, d_n d'_n, d_1 v'_1 + v_1 d'_2, \dots, d_{n-1} v'_{n-1} + v_{n-1} d'_n, v_n t \alpha(d'_1) + d_n v'_n t'_n] \\ &= [d_1, \dots, d_n, v_1, \dots, v_n t][d'_1, \dots, d'_n, v'_1, \dots, v'_n t] \\ &= g([d_1, \dots, d_n, v_1, \dots, v_n])g([d'_1, \dots, d'_n, v'_1, \dots, v'_n]) \end{aligned}$$

which shows that  $g$  is an isomorphism of rings. The proof is thereby complete.

■

LEMMA 2.1: *Let  $A$  be a right self injective ring in which every idempotent is central and let  $M$  be an injective simple right  $A$ -module. Then there exists an idempotent  $v \in A$  such that  $Mv = M$  and  $vA$  is a strongly regular right self injective ring.*

*Proof:* Let  $J(A)$  denotes the Jacobson radical of  $A$ . Set  $\bar{A} = A/J(A)$  and let  $a \mapsto \bar{a}$ ,  $a \in A$ , be the canonical projection of rings  $A \rightarrow \bar{A}$ . It follows from [15, Corollary 4.10 and Theorem 4.7] that every idempotent of  $\bar{A}$  is of the form  $\bar{u}$  where  $u = u^2 \in A$ . Therefore idempotents in  $A$  are central and

$$(12) \quad \bar{A} \text{ is a right self injective strongly regular ring.}$$

It follows from Zorn's lemma that there exists a family  $\{x_i \mid i \in I\}$  of elements of  $J(A)$  maximal with respect to the property  $\sum_{i \in I} x_i A = \bigoplus_{i \in I} x_i A$ . Set  $K = \bigoplus_{i \in I} x_i A$ . Let  $wA$  be an injective hull of  $K$  where  $w = w^2 \in A$ . Pick idempotents  $w_i \in wA$ ,  $i \in I$ , such that each  $w_i A$  is an injective hull of  $x_i A$ . Since idempotents in  $A$  are central and  $x_i \in w_i A$ , we have

$$(13) \quad ww_i - w_i = w_i w \quad \text{and} \quad w_i x_i = x_i = x_i w_i \quad \text{for all } i \in I.$$

As  $\sum_{i \in I} x_i A = \bigoplus_{i \in I} x_i A$ , we conclude that  $\sum_{i \in I} w_i A = \bigoplus_{i \in I} w_i A$  and so

$$(14) \quad \{w_i \mid i \in I\} \text{ is a family of pairwise orthogonal central idempotents of } A.$$

Obviously

$$(15) \quad wA \text{ is an injective hull of } \bigoplus_{i \in I} w_i A.$$

We claim that

$$(16) \quad wJ(A) = J(A).$$

It is enough to show that  $(1 - w)J(A) = 0$ . Assume to the contrary that  $(1 - w)J(A) \neq 0$  and pick  $0 \neq y \in (1 - w)J(A)$ . Clearly  $yA \cap wA = 0$  and whence  $yA \cap \bigoplus_{i \in I} x_iA = 0$ , a contradiction to the choice of the family  $\{x_i \mid i \in I\}$ . Therefore (16) is proved.

We define a homomorphism of right  $A$ -modules  $f: \bigoplus_{i \in I} w_iA \rightarrow \bigoplus_{i \in I} x_iA$  by the rule  $w_i \rightarrow x_iw_i$ ,  $i \in I$ . Since  $w_i$  is central, we see that  $f(w_i) = w_ix_i$ ,  $i \in I$ . Recalling that  $wA$  is an injective right  $A$ -module containing both  $\bigoplus_{i \in I} w_iA$  and  $\bigoplus_{i \in I} x_iA$ , we conclude that  $f$  can be extended to an endomorphism of  $wA_A$ . Obviously  $\text{End}(wA_A) = wAw$  and so there exists  $x \in wAw$  such that  $f(y) = xy$  for all  $y \in \bigoplus_{i \in I} w_iA$ . In particular,  $w_ix_i = f(w_i) = xw_i = w_ix$  and because  $x_iA \subseteq w_iA$ , we have that

$$(17) \quad w_ix = w_ix_i = x_i \quad \text{for all } i \in I.$$

We claim that

$$(18) \quad x \in J(A), \quad \text{and} \quad wA \text{ is an injective hull of } xA.$$

Indeed, assume to the contrary that  $x \notin J(A)$ . Clearly  $\bar{x}$  is a nonzero element of the ring  $\bar{A}$  and so (12) implies that there exists  $y \in A$  such that  $\overline{xy}$  is a nonzero idempotent. It is well-known that there exists a nonzero idempotent  $u \in A$  with  $xy - u \in J(A)$  (see [15, Corollary 4.10 and Theorem 4.7]). Since  $x \in wA$ ,  $(1 - w)x = 0$  and so  $(1 - w)u \in J(A)$ . Recalling that idempotents in  $A$  are central, we see that  $(1 - w)u = 0$  because the Jacobson radical of a ring does not contain nonzero idempotents. Therefore  $wu = u$  and whence  $0 \neq uA \subseteq wA$ . Next,  $J(A) \ni w_i(xy - u) = (w_ix)y - w_iu = x_iy - w_iu$  by (17) and so  $w_iu \in J(A)$  (because  $x_i \in J(A)$ ) forcing  $w_iu = 0$  for all  $i \in I$ . Since  $\bigoplus_{i \in I} x_iA$  is an essential submodule of  $wA$  and  $0 \neq uA \subseteq wA$ , there exist a nonempty finite subset  $I_0 \subseteq I$  and elements  $a_i \in A$ ,  $i \in I_0$ , such that  $0 \neq \sum_{i \in I_0} x_ia_i \in uA \cap \bigoplus_{i \in I} x_iA$ . Pick  $j \in I_0$  with  $x_ja_j \neq 0$ . Then  $0 \neq x_ja_j = w_j \sum_{i \in I_0} x_ia_i \in w_juA = 0$ , a contradiction. Therefore  $x \in J(A)$ . Finally, by (17),  $xA \supseteq xw_iA = w_ixA = x_iA$  for all  $i \in I$  and so  $xA \supseteq K$ . We see that  $xA$  is an essential submodule of  $wA$  and whence (18) is proved.

Now let  $P = r(A; M)$ . Since  $M$  is a simple right  $A$ -module,  $P$  is a primitive right ideal of  $A$ . Let  $B = \{b \in A \mid b^2 = b \in P\}$  and let  $Q = \sum_{b \in B} bA$ . Clearly  $Q \subseteq P$ . We claim that

$$(19) \quad P = J(A) + Q.$$

Indeed, the inclusion  $P \supseteq J(A) + Q$  is obvious. Therefore it is enough to show that  $\bar{P} \subseteq \bar{Q}$ . Every ideal of a regular ring is generated by idempotents. Given an idempotent  $\bar{u} \in \bar{P}$ ,  $u \in P$ , by [15, Corollary 4.10 and Theorem 4.7] there exists an idempotent  $a \in A$  with  $u - a \in J(A) \subseteq P$  (i.e.,  $\bar{u} = \bar{a}$ ) and so  $a \in P$  forcing  $a \in B$  and whence (19) is proved.

Setting  $A' = A/Q$ , we denote by  $a \mapsto a'$  the canonical projection of rings  $A \rightarrow A'$ . Obviously  $P' = J(A)'$  forcing  $P' \subseteq J(A')$ . Clearly  $A'/P' \cong A/P \cong \bar{A}/\bar{P}$ . We see that  $\bar{A}/\bar{P}$  is a skew field because  $\bar{P}$  is a primitive right ideal of the strongly regular ring  $\bar{A}$  (see (12)). Therefore  $A'/P'$  is a skew field as well. We conclude that

$$(20) \quad A' \text{ is a local ring with maximal ideal } P' = J(A').$$

Since  $Q \subseteq P = r(A; M)$ ,  $M$  is  $A'$ -module canonically. By assumption  $M$  is an injective simple right  $A$ -module. Therefore  $M$  is an injective simple right  $A'$ -module. Since over a local ring simple modules are isomorphic, we conclude from (20) that  $A'$  is a right  $V$ -ring. Therefore  $J(A') = 0$  by [16, 23.1(i)] forcing  $P = Q$ . By (18),  $x \in J(A) \subseteq P = Q$  and whence there exist  $b_1, b_2, \dots, b_n \in B$  and  $a_1, a_2, \dots, a_n \in A$  such that  $x = \sum_{i=1}^n b_i a_i$ . Since each  $b_i \in P$ ,  $1 - b_i \notin P$ . Recalling that  $P$  is a prime ideal of  $A$  and idempotents in  $A$  are central, we conclude that  $v = (1 - b_1)(1 - b_2) \dots (1 - b_n) \notin P$ . Therefore  $1 - v \in P$  forcing  $M(1 - v) = 0$  and so  $Mv = M$ . Next, clearly  $vb_i = 0$  for all  $t = 1, 2, \dots, n$  and whence  $vx = \sum_{i=1}^n vb_i a_i = 0$ . Therefore  $vA \cap xA = 0$ . It now follows from (18) that  $vA \cap vA = 0$  and whence (16) implies that  $vA \cap J(A) = 0$ . We see that  $vA \cong (vA)/(vA \cap J(A)) = \bar{v}\bar{A}$  is a right self injective strongly regular ring because  $\bar{A}$  is so by (12). The proof is complete. ■

*Remark 2.2:* A ring  $R = G(n; \Delta; P)$  is not left self injective.

*Proof:* Given an integer  $1 \leq i \leq n$ , we denote by  $e_i$  the matrix in  $R$  whose  $(i, i)$  entry is 1 and all other entries zero. Set  $e = e_1$  and  $L = e_1 R e_2$ . Clearly  $e R e \cong \Delta/P = D$  is a skew field and  $\dim({}_e R e L) = 1$ . Choose any  $0 \neq p \in L$  and note that  $e R e p = L$ . Define a map  $f: L \rightarrow e R e$  by the rule  $(xp)f = x$  for all  $x \in e R e$ . Clearly  $f$  is a well-defined additive map. We claim that  $f$  is a homomorphism of left  $R$ -modules. To this end, note that  $R e = e R e$ . Take  $y \in L$  and  $z \in R$ . Clearly  $y = xp$  for some  $x \in e R e$ . Since  $cy = y$ ,  $zy = (ze)y$ . Recalling that  $R e = e R e$ , we see that  $ze = eze$ . Therefore  $zy = (eze)e p$  and so

$$\begin{aligned} (zy)f &= ((eze)e p)f = eze e = (eze)(e p) = (ze)(e p) \\ &= z(e p) = z[(e p)f] = z(yf) \end{aligned}$$

which proves our claim. Since  ${}_R R$  is injective, there exists  $r \in R$  such that  $yf = yr$  for all  $y \in L$ . As  $ye_2 = y$  for all  $y \in L$ , we may assume without loss of generality that  $r \in e_2 R$ . As  $e_2 R = e_2 R e_2 + e_2 R e_3$ , we see that  $re = re_1 = 0$ , contradicting  $lf \in eR$ . The proof is thereby complete. ■

*Remark 2.3:* Let  $A$  be a ring such that the following conditions are fulfilled:

- (1) Every right ideal of  $A$  is two sided.
- (2) For any  $a \in A$  and any  $A$ -module map  $f: aA \rightarrow A$  there exists an element  $b \in A$  such that  $fx = xb$  for all  $x \in aA$ .

Then every left ideal of  $A$  is two sided.

*Proof:* It is enough to show that  $Aa$  is an ideal of  $A$  for any  $a \in A$ . Given a nonempty subset  $S \subseteq A$ , we set

$$\ell(S) = \{x \in A \mid xS = 0\} \quad \text{and} \quad r(S) = \{x \in A \mid Sx = 0\}.$$

We claim that  $\ell(r(Aa)) = Aa$ . Indeed,  $Aa \subseteq \ell(r(Aa))$ . Let  $b \in \ell(r(Aa))$ . Then  $bx = 0$  for all  $x \in A$  with  $ax = 0$ . Therefore the map  $f: aA \rightarrow bA$ ,  $ax \mapsto bx$ , is well defined and so there exists an element  $c \in A$  with  $fy = cy$  for all  $y \in aA$ . In particular,  $b = fa = ac$  which proves our claim. Finally,  $r(Aa)$  is a right ideal and so it is an ideal of  $A$ . Therefore  $Aa = \ell(r(Aa))$  is an ideal of  $A$  as well. The proof is completed. ■

Incidentally, an immediate consequence of the above remark is that every right duo right principally injective ring is left duo.

*Proof of Theorem 1.5:* Making use of Theorems 1.2 and 1.3, we obtain that every right  $q$ -ring  $R$  is isomorphic to a finite direct product of rings of the following kinds:

- (1) Semisimple Artinian ring.
- (2)  $H(n; D; \alpha)$ , where  $\alpha$  is an automorphism of  $D$ .
- (3)  $G(n; \Delta; P)$ , where  $\Delta$  is a right  $q$ -ring all of whose idempotents are central.
- (4) A right  $q$ -ring all of whose idempotents are central.

Now consider a ring  $G(n; \Delta; P)$ . Since  $M = \Delta/P$  is an injective  $\Delta$ -module, Lemma 2.1 implies that there exists an idempotent  $v \in \Delta$  such that  $Mv = M$  and  $v\Delta$  is a strongly regular right self injective ring. The first part of Theorem 1.5 now follows from obvious isomorphism  $G(n; \Delta; P) \cong G(n; v\Delta; vP) \times (1-v)\Delta$  together with Remark 2.3.

To prove the last statement of the theorem, we now assume that  $R$  is left self injective. It follows from both Remark 2.2 and the first part of the theorem that  $R$  is isomorphic to a finite direct product of rings of the following kinds:

- (1) Semisimple Artinian ring.
- (2)  $H(n; D; \alpha)$ , where  $\alpha$  is an automorphism of  $D$ .
- (3) A right  $q$ -ring all of whose idempotents are central.

Clearly every semisimple artinian ring is both a left and a right  $q$ -ring. According to [8, Corollary 3.12] every ring of the form  $H(n; D; \alpha)$  is a left  $q$ -ring. Finally, it follows from both Theorem 1.1 and Remark 2.3 that every left injective right  $q$ -ring is also a left  $q$ -ring. The proof is thereby complete.

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