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ON THE SYMMETRY OF THE GOLDIE AND CS CONDITIONS FOR PRIME RINGS

DINH VAN HUYNH, S. K. JAIN, AND S. R. LÓPEZ-PERMOUTH

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ABSTRACT. It is shown that: (a) If R is a prime right Goldie right CS ring with right uniform dimension at least 2, then R is left Goldie, left CS; (b) A semiprime ring R is right Goldie left CS iff R is left Goldie, right CS.

All rings are associative having an identity and all modules are unitary. A right module M over a ring R is called CS (or extending) if every submodule of M is essential in a direct summand of M, or equivalently, if every complement submodule of M is a direct summand of M. A ring R is called right CS (resp., left CS), if R_R (resp., ${}_RR$) is a CS module. CS modules have been extensively studied by many authors.

A ring R is defined to be a right (left) Goldie ring if R has ascending chain condition on right (left) annihilators and the right (left) uniform dimension of R is finite. A right Goldie ring R is (semi-)prime if and only if R has classical right quotient ring which is (semi-)simple artinian. For notation not defined here we refer the reader to [1], [2] and [3].

Theorem 1. *A prime right Goldie, right CS ring R with right uniform dimension at least 2, is left Goldie, and left CS.*

Proof. Let n be the right uniform dimension of R. By assumption, $n \geq 2$. Since R is right CS, $R = e_1R \oplus \cdots \oplus e_nR$ where each e_iR is uniform and $\{e_i\}_{i=1}^n$ is a system of orthogonal idempotents of R. Let Q be the classical right quotient ring of R. Then we have:

$$R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 & \cdots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \cdots & e_2Re_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_nRe_1 & e_nRe_2 & \cdots & e_nRe_n \end{pmatrix} \subseteq \begin{pmatrix} e_1Qe_1 & e_1Qe_2 & \cdots & e_1Qe_n \\ e_2Qe_1 & e_2Qe_2 & \cdots & e_2Qe_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_nQe_1 & e_nQe_2 & \cdots & e_nQe_n \end{pmatrix} \cong Q.$$

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Henceforth, we will identify

$$\begin{pmatrix} e_1Re_1 & e_1Re_2 & \cdots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \cdots & e_2Re_n \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ e_nRe_1 & e_nRe_2 & \cdots & e_nRe_n \end{pmatrix}$$

with R and

$$\begin{pmatrix} e_1Qe_1 & e_1Qe_2 & \cdots & e_1Qe_n \\ e_2Qe_1 & e_2Qe_2 & \cdots & e_2Qe_n \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ e_nQe_1 & e_nQe_2 & \cdots & e_nQe_n \end{pmatrix}$$

with Q.

Let $\alpha = \begin{pmatrix} a_1 & \cdots & . \\ a_2 & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ a_n & \cdots & . \end{pmatrix}$ be an element of Q with $a_i \in e_iQe_1$, $a_1 \neq 0$. Then for the minimal right ideal $M = \begin{pmatrix} e_1Qe_1 & e_1Qe_2 & \cdots & e_1Qe_n \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ of Q, αM is a minimal right ideal of Q, too. Hence $R \cap \alpha M$ is a (nonzero) closed uniform right ideal of R. Consequently, $R \cap \alpha M$ is generated by an idempotent $e \in R$. Therefore, there exists an element $\beta = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ such that $\alpha\beta = \begin{pmatrix} a_1x_1 & a_1x_2 & \cdots & a_1x_n \\ a_2x_1 & a_2x_2 & \cdots & a_2x_n \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_nx_1 & a_nx_2 & \cdots & a_nx_n \end{pmatrix} = e \in R$, where $x_i \in e_1Qe_i$. Hence $a_ix_j \in e_iRe_j$ for $i, j = 1, \cdots, n$. Note that at least one x_i is nonzero. After squaring this matrix and comparing the corresponding entries of this matrix and its square we get $(a_1x_k)(a_kx_k) + \cdots + (a_1x_n)(a_nx_k) = a_1x_k$, where x_k is the first nonzero entry in the first row of β. Since $a_1 \neq 0$, it follows $x_ka_kx_k + x_{k+1}a_{k+1}x_k + \cdots + x_na_nx_k = x_k$. As $x_k \in e_1Qe_k$, there exists $x_k^* \in e_kQe_1$ such that $x_kx_k^* = e_1$, because $e_1Q \cong e_kQ$.

Consequently,

$$(1) \qquad x_k a_k + x_{k+1} a_{k+1} + \cdots + x_n a_n = e_1.$$

Note that if e_1Re_1 is a division ring, then R is a simple artinian ring. In this case the statement in our theorem is trivially true. Therefore, we assume that $e_1Re_1 \neq e_1Qe_1$. Since $n > 1$, we can choose $a_1 \in e_1Qe_1 \setminus e_1Re_1$, $a_2 = \cdots = a_{n-1} = 0$, $0 \neq a_n \in e_nRe_1$. Then we must have $k = 1$, i.e. $x_1 \neq 0$. For, if $x_1 = 0$, the equation (1) becomes $x_n a_n = e_1$. Multiplying this with a_1 on the left we get $(a_1 x_n)a_n = a_1 e_1 = a_1$. From this and $a_1 x_n \in e_1Re_n$, it follows that $a_1 \in e_1Re_1$, a contradiction. Hence $x_1 \neq 0$. Thus equation (1) becomes $x_1a_1 + x_na_n = e_1$, and so

$$(2) \qquad x_1 a_1 = e_1 - x_n a_n.$$

Let $0 \neq y \in e_1Re_n \subset e_1Qe_n$. So there exists $y' \in e_nQe_1$, such that $y'y = e_n$, because $e_1Q \cong e_nQ$. We note that $ya_n \neq 0$. For, if $ya_n = 0$, then $y'(ya_n) = (y'y)a_n = 0$, and so $a_n = 0$, a contradiction. Therefore $(ya_n)x_1 \neq 0$. Now $0 \neq y(a_nx_1) \in (e_1Re_n)(e_nRe_1) \subseteq e_1Re_1$. Furthermore, $ya_nx_n \in e_1Re_n$, and $a_n \in e_nRe_1$ yield $ya_nx_na_n \in e_1Re_1$. Next, multiplying (2) on the left by ya_n we get $(ya_nx_1)a_1 = ya_n - ya_nx_na_n$. Consequently, $a_1 = (ya_nx_1)^{-1}(ya_n - ya_nx_na_n)$. This shows that e_1Re_1 is a left Ore domain. Similarly, we conclude that every e_iRe_i is a left Ore domain.

Therefore, each Re_i is a uniform left ideal of R. This is folklore; however we provide an argument here for the sake of completeness: If A, B are nonzero submodules of Re_i such that $A \cap B = 0$, then $e_iA \cap e_iB = 0$. Since e_iA and e_iB are left ideals of the left Ore domain e_iRe_i, either $e_iA = 0$ or $e_iB = 0$. Consequently, either $BA = 0$ or $AB = 0$. This is a contradiction because R is a prime ring, proving the claim. Since $R = Re_1 \oplus \cdots \oplus Re_n$, ${}_RR$ has finite uniform dimension. Moreover, as R is prime right Goldie, it has DCC on right annihilators (cf. [4, Lemma 7.2.2]). Therefore, R has ACC on left annihilators, proving that R is left Goldie.

Finally, we show that R is left CS. Note that Q is the classical left and right quotient ring of R. Let U be a non-essential left ideal of R. Then there are orthogonal idempotents e, $f \in Q$, such that $Q = Qe \oplus Qf$, where U is essential in ${}_RQe$, and $Qf \neq 0$. Hence $U(fQ) = 0$. As $fQ \cap R \neq 0$, the right annihilator of U in R is nonzero. Moreover, let $0 \neq a \in R$ and $r(a)$ be the right annihilator of a in R. Then $R = C \oplus D$, where $r(a)$ is essential in D. Since $aR \cong R/r(a)$ and aR is a nonsingular right ideal of R, we must have $r(a) = D$. This shows that $aR \cong C$, and in particular that aR is projective. Hence R is a right p.p. ring. Thus R is left CS by [1, Proposition 12.3]. The proof is complete. □

Remark 1. Theorem 1 is not true, in general, if the right uniform dimension of the prime right Goldie ring is 1, since there exist right Ore domains (hence right CS) which are not left Ore (hence not left CS). For the existence of such a domain, see [3, Exercise 1, p. 101].

Remark 2. Let R be a semiprime right Goldie right CS ring. Then R_R is a direct sum of uniform right ideals e_iR, $i = 1, \cdots, n$, $e_i^2 = e_i$. After renumbering the indices, if necessary, we get $R = [e_1R] \oplus \cdots \oplus [e_tR]$, where each $[e_jR]$ is a direct sum of uniform right ideals belonging to $\{e_iR\}_{i=1}^n$ that are subisomorphic to each other, and $Hom_R(e_jR, e_kR) = 0$ for $j \neq k$ ($j, k \in \{1, \cdots, t\}$). It is easy to check that each $R_j = [e_jR]$ is an ideal of R, and is itself a prime right Goldie right CS ring. Hence $R = R_1 \oplus \cdots \oplus R_t$ is a ring direct sum of prime right Goldie right

CS rings. Let n_j be the right uniform dimension of R_j. By Theorem 1, for any $n_j > 1$, R_j is also left Goldie and left CS.

The following consequence of Theorem 1 is a stronger version of [1, Corollary 12.9].

Corollary 2. *For a domain K the following conditions are equivalent:*
(a) $(K \oplus K)_K$ *is CS;*
(b) ${}_K(K \oplus K)$ *is CS.*
If K satisfies (a) or (b), then K is right and left Ore.

Proof. $(a) \Rightarrow (b)$. By (a), K is right Ore; hence the 2×2 matrix ring $M_2(K)$ over K is a prime right Goldie ring of right uniform dimension 2. Moreover, by [1, Lemma 12.8], (a) implies that $M_2(K)$ is right CS. By Theorem 1, $M_2(K)$ is left CS. Again by [1, Lemma 12.8], ${}_K(K \oplus K)$ is CS, proving (b). Similarly $(b) \Rightarrow (a)$ holds. The last statement is clear. □

Theorem 3. *For a semiprime ring R, the following conditions are equivalent:*
(i) R is left Goldie, right CS;
(ii) R is right Goldie, left CS.
In this case, $R = R_1 \oplus \cdots \oplus R_n$, *where each* R_i *is prime, right Goldie, left Goldie, right CS and left CS.*

Proof. We need only show $(i) \Rightarrow (ii)$; then the implication $(ii) \Rightarrow (i)$ is obtained in a similar way.

Let R be a semiprime left Goldie right CS ring. We claim that R has finite right uniform dimension. Assume on the contrary, that R contains an infinite direct sum $\bigoplus_{i=1}^{\infty} A_i$ of nonzero right ideals A_i. Let K_1 be the complement of A_1 in R containing $\bigoplus_{i=2}^{\infty} A_i$. Since R is right CS, $R = K_1 \oplus B_1$ for some nonzero right ideal B_1 of R. Let K_2 be the complement of A_2 in K_1 containing $\bigoplus_{i=3}^{\infty} A_i$. Since $(K_1)_R$ is CS, $K_1 = K_2 \oplus B_2$ for some nonzero submodule B_2 of K_1. This yields $R = K_2 \oplus B_1 \oplus B_2$. Proceeding in this way we can produce an arbitrary number of orthogonal idempotents in R, a contradiction, because R is left Goldie. Hence R has finite right uniform dimension. Since R is semiprime left Goldie, R has DCC on left annihilators, and so R is right Goldie. By Remark 2, $R = R_1 \oplus \cdots \oplus R_t$, a direct sum of prime right and left Goldie right CS rings. Let $n_i = \text{u-dim}(R_i)_{R_i} = \text{u-dim}({}_{R_i}R_i)$. If $n_i = 1$, then R_i is a uniform left R_i-module, and hence left CS. For $n_i \geq 2$ we apply Theorem 1 to obtain that R_i is also left CS. Hence R is left CS. The last statement is clear from the proof. □

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Institute of Mathematics, P.O. Box 631 Boho, Hanoi, Vietnam - Department of Mathematics, Ohio University, Athens, Ohio 45701

Department of Mathematics, Ohio University, Athens, Ohio 45701

Department of Mathematics, Ohio University, Athens, Ohio 45701