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EXTENSIONS OF $\mathcal{G}$-BASED MATRIX PARTIAL ORDERS*

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Abstract. We prove that a partial order $\preceq^{\mathcal{G}}$ on $\mathbf{R}^{m\times n}$ can always be extended to a $\mathcal{G}$-based matrix partial order $\preceq^{\mathcal{G}^*}$ such that $\mathcal{G}^*(A) \neq \emptyset$ for all $A \in \mathbf{R}^{m\times n}$, thus answering an open question [Mitra, *Linear Algebra Appl.*, 148 (1991), pp. 237–263]. It is further shown that this result does not in general remain true if besides $\mathcal{G}$, we also insist that $\mathcal{G}^*$ be semicomplete. And even if in a special situation this is possible and if card $\mathcal{G}(A) \leq 1$ for each A, this does not mean that there also need be a semicomplete extension such that $\mathcal{G}^*(A)$ is a singleton for all A. In addition, some other interesting results on matrix partial orders are given. For instance, a useful characterization for a semicomplete map to induce a partial order on the set of square matrices is derived.

Key words. $\mathcal{G}$-based matrix partial order, star order, minus order, sharp order, semicomplete map, property-p map

AMS subject classifications. 15A30, 15A09

1. Introduction. This paper continues a series of recent articles investigating different types of matrix orders and discussing their properties and relations; see [6], [7], [8], [18]. To facilitate reading, we first present the order concept, which is of interest to us in this paper.

Let $\mathbf{R}^{m\times n}$ denote the set of real $m \times n$ matrices, and let $\mathcal{P}(S)$ denote the power set of a set S. Moreover, let

$$\mathcal{G}: \quad \mathbf{R}^{m\times n} \rightarrow \mathcal{P}(\mathbf{R}^{n\times m})$$

be a map such that

$$\mathcal{G}(A) \subseteq A\{1\}, \tag{1.1}$$

where $A\{1\}$ denotes the set of all g-inverses of A; see §2. The map $\mathcal{G}$ is called *semicomplete* if for every matrix $A \in \mathbf{R}^{m\times n}$ one has $GAG \in \mathcal{G}(A)$ whenever $G \in \mathcal{G}(A)$. Define the relation $\preceq^{\mathcal{G}}$ on $\mathbf{R}^{m\times n}$ by saying

$$A \preceq^{\mathcal{G}} B \quad \text{if} \quad (B-A)X = 0,\ X(B-A) = 0 \quad \text{for some} \quad X \in \mathcal{G}(A). \tag{1.2}$$

This relation is said to be a $\mathcal{G}$*-based relation*, and if $\mathcal{G}$ is semicomplete, then $\preceq^{\mathcal{G}}$ is also said to be *semicomplete*. Call the set

$$\Omega_{\mathcal{G}} := \{A \in \mathbf{R}^{m\times n} \mid \mathcal{G}(A) \neq \emptyset\}$$

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the *support* of $\mathcal{G}$ or $\preceq^{\mathcal{G}}$. The relation $\preceq^{\mathcal{G}}$ is automatically *antisymmetric*. Furthermore, it is trivially *reflexive on its support*. Throughout this paper, a $\mathcal{G}$-based relation is therefore said to be a *partial order* (on $\mathbf{R}^{m\times n}$) if it is *transitive*. Since in the literature the notion of a partial order is sometimes defined in a stronger manner, it is pertinent to emphasize here that in our definition the condition of reflexivity is not required to hold on the whole of $\mathbf{R}^{m\times n}$. That $\preceq^{\mathcal{G}}$ as defined by (1.2) need not always correspond to a partial order, that is, that $\preceq^{\mathcal{G}}$ need not necessarily be transitive, is illustrated by Example 1 in [6, p. 242]. Sufficient conditions that make $\preceq^{\mathcal{G}}$ a partial order are studied in [6]. Observe that the above $\mathcal{G}$-based relation concept, which is due to Mitra [6], covers as special cases the various known matrix orders such as the *star order* $\preceq^{*}$ (set $\mathcal{G}(A) = \{A^{\dagger}\}$), the *minus order* $\preceq^{-}$ (set $\mathcal{G}(A) = A\{1\}$), and the *sharp order* $\preceq^{\#}$ (set $\mathcal{G}(A) = \{A^{\#}\}$ if A has index 1, and set $\mathcal{G}(A) = \emptyset$ otherwise), to mention only a few. Next, let $\preceq^{\mathcal{G}}$ correspond to a partial order. The sharp order provides us with a typical $\mathcal{G}$-based partial order that does not support the whole set of square matrices. For such a case the following question was an interesting open problem (see [6, p. 252]): Is it possible to modify $\mathcal{G}$ only on the complement of $\Omega_{\mathcal{G}}$ so that the modified map, say $\mathcal{G}^*$, continues to induce a partial order? Let us call a map $\mathcal{G}^*$ which is defined like $\mathcal{G}$ an *extension* of $\mathcal{G}$ if $\mathcal{G}^*(A) = \mathcal{G}(A)$ for each $A \in \Omega_{\mathcal{G}}$, and if $\mathcal{G}^*$ is an extension of $\mathcal{G}$, then let us call the induced relation $\preceq^{\mathcal{G}^*}$ a *$\mathcal{G}$-based extension* of the relation $\preceq^{\mathcal{G}}$. Of particular interest to us is the question of whether a partial order $\preceq^{\mathcal{G}}$ with $\Omega_{\mathcal{G}} \neq \mathbf{R}^{m\times n}$ admits a $\mathcal{G}$-based partial order extension $\preceq^{\mathcal{G}^*}$ such that $\mathcal{G}^*(A) \neq \emptyset$ for all A. This question was affirmatively answered in the special case of the sharp order in [7].

The purpose of this paper is manifold. In §4 we prove that the above latter question can, in general, be answered in the affirmative. Actually, we show that there do exist many $\mathcal{G}$-based extensions. When along with $\preceq^{\mathcal{G}}$ its extension $\preceq^{\mathcal{G}^*}$ is also required to be semicomplete, the answer becomes more complicated; this case is discussed in §5. A typical example of a semicomplete $\mathcal{G}$-based partial order $\preceq^{\mathcal{G}}$ not allowing such an extension is given there. Of course, this does not mean that it is always impossible. But even if for a certain $\preceq^{\mathcal{G}}$ there is such a semicomplete partial order extension, this does not mean that there also need be such an extension $\preceq^{\mathcal{G}^*}$ for which $\mathcal{G}^*(A)$ is a singleton set for all $A \notin \Omega_{\mathcal{G}}$. The sharp order (see Example 4.3 along with Theorem 5.5) can serve as an example to illustrate this fact. In addition, some further interesting results on matrix partial orders are given. Section 3, for instance, contains a useful characterization for a semicomplete map $\mathcal{G}$, which is defined on the set of square $n\times n$ matrices, to induce a partial order $\preceq^{\mathcal{G}}$. The concept of a *property-p relation* discussed in §5 arises from this characterization. It was implicitly studied earlier by Mitra in [6], although its significance is brought out in the present paper. It is shown in §5 that such a relation always possesses a (unique) *maximal* property-p extension, and a method for getting this extension is given. Section 2 contains a few background results from the theory of g-inversion which are used in the subsequent sections and which might also be very useful in a future attempt to solve the open problem stated in §5.

2. Generalized inversion and preliminaries. Let $\mathcal{M}$ and $\mathcal{N}$ be linear subspaces in the n-dimensional real space $\mathbf{R}^n$. Then $\mathcal{M}^{\perp}$ will denote the orthogonal complement of $\mathcal{M}$ in $\mathbf{R}^n$ (with respect to the usual inner product), and if $\mathcal{M}\cap\mathcal{N} = \{0\}$, then $\mathcal{M}\oplus\mathcal{N}$ will denote the direct sum of $\mathcal{M}$ and $\mathcal{N}$. Next, if $\mathcal{N}$ is a direct complement of $\mathcal{M}$ (i.e., $\mathbf{R}^n = \mathcal{M}\oplus\mathcal{N}$), then $P_{\mathcal{M},\mathcal{N}}$ will denote the well-defined projector on $\mathcal{M}$ along $\mathcal{N}$ (see, e.g., [10, pp. 106–113]). Notice that $P_{\mathcal{M},\mathcal{N}}$ may be defined by $P_{\mathcal{M},\mathcal{N}}u = u$ if $u \in \mathcal{M}$ and $P_{\mathcal{M},\mathcal{N}}u = 0$ if $u \in \mathcal{N}$. For a given matrix A in the space

$\mathbf{R}^{m\times n}$ of all real $m \times n$ matrices, denote by A^t, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}_c(A)$, and $\mathcal{R}_c(A)$, respectively, the transpose of A, the null space of A, the range space of A, the set of all direct complements of $\mathcal{N}(A)$, and the set of all direct complements of $\mathcal{R}(A)$. Let I be the identity matrix and 0 the zero matrix of whatever size is appropriate for the context. Further, we denote by $A\mathcal{M}$ the image of $\mathcal{M}$ under A; i.e., $A\mathcal{M} = \{Au \mid u \in \mathcal{M}\}$.

Now let $A \in \mathbf{R}^{m\times n}$, let $\mathcal{M} \in \mathcal{N}_c(A)$, and let $\mathcal{S} \in \mathcal{R}_c(A)$. Consider the matrix equations

$$\text{(2.1)} \qquad \begin{array}{llll} \text{(G1)} & AXA = A, & (\text{G}\mathcal{M}) & XA = P_{\mathcal{M},\mathcal{N}(A)}, \\ \text{(G2)} & XAX = X, & (\text{G}\mathcal{S}) & AX = P_{\mathcal{R}(A),\mathcal{S}}. \end{array}$$

Suppose that $\emptyset \neq \eta \subseteq \{1, 2, \mathcal{M}, \mathcal{S}\}$. Then let $A\eta$ denote the set of all those matrices X that satisfy equations (Gi) for all $i \in \eta$. Any $X \in A\eta$ is called an *η-inverse* of A, and is denoted by A^{η}. $\{1\}$-inverses are usually called *generalized inverses* or *g-inverses* and are also denoted by A^-. $\{1,2\}$-inverses are called *reflexive g-inverses* and are also denoted by A_r^-. For an extensive discussion of the theory of g-inversion, we refer, e.g., to the books by Ben-Israel and Greville [1], Hartung and Werner [3], Pringle and Rayner [9], and Rao and Mitra [10]; for a geometric approach, to Werner [12, Chap. 1] and Rao and Yanai [11]; and for a projector theoretical one to, e.g., the paper by Langenhop [4].

Only for the sake of clarity and for easier reference do we mention the following well-known results (cf. [12]; see also [15], [17]).

THEOREM 2.1. *For given $A \in \mathbf{R}^{m\times n}$, we have the following.*

(i) *The $\{2, \mathcal{M}, \mathcal{S}\}$-inverse of A exists uniquely.*

(ii) *Any $\{\mathcal{M}\}$-inverse of A and likewise any $\{\mathcal{S}\}$-inverse of A is always a $\{1\}$-inverse of A. Conversely, for any $\{1\}$-inverse of A there uniquely exist an $\mathcal{M} \in \mathcal{N}_c(A)$ and an $\mathcal{S} \in \mathcal{R}_c(A)$ such that $X \in A\{\mathcal{M}, \mathcal{S}\}$. Moreover, if $X \in A\{\mathcal{M}, \mathcal{S}\}$, then $XAX = A^{\{2,\mathcal{M},\mathcal{S}\}}$.*

(iii) *If $X \in A\{\mathcal{M}, \mathcal{S}\}$, then $\mathcal{M} = \mathcal{R}(XA) \subseteq \mathcal{R}(X)$, $\mathcal{N}(X) \subseteq \mathcal{S}$, and $X\mathcal{S} \subseteq \mathcal{N}(A)$. Hence, in particular,* $\operatorname{rank}(A) = \operatorname{rank}(AX) = \operatorname{rank}(XA)$. *Moreover, $X = A^{\{2,\mathcal{M},\mathcal{S}\}}$ iff $\mathcal{R}(X) = \mathcal{M}$ and $\mathcal{N}(X) = \mathcal{S}$.*

(iv) *If* $\operatorname{rank}(A) = r < \min\{m, n\}$ *then, for each s with $r \leq s \leq \min\{m, n\}$, there exist g-inverses A^- such that* $\operatorname{rank}(A^-) = s$. *Moreover,* $\operatorname{rank}(A) = \operatorname{rank}(A^-)$ *iff A^- is a reflexive g-inverse.*

(v) *The $\{2, \mathcal{R}(A^t), \mathcal{N}(A^t)\}$-inverse of A coincides with the Moore–Penrose inverse of A and is henceforth denoted by $A^\dagger$.*

When A is square, $\operatorname{ind}(A)$, the *index* of A, denotes the smallest positive integer k for which $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ or, equivalently, $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. Now, let A be any square matrix of index 1. Then $\mathcal{R}(A) \in \mathcal{N}_c(A)$ and $\mathcal{N}(A) \in \mathcal{R}_c(A)$, so that by Theorem 2.1(i) the $\{2, \mathcal{R}(A), \mathcal{N}(A)\}$-inverse exists and is unique. This g-inverse is called the *group-inverse* of A and is denoted by $A^\#$. It is the unique $\{1,2\}$-inverse X of A satisfying $AX = XA$. Recall that a square matrix A has $\operatorname{ind}(A) = k$ iff there exist a *nilpotent* matrix N_A of degree k (i.e., $N_A^k = 0$ whereas $N_A^{k-1} \neq 0$) and a *core* matrix C_A (i.e., $\operatorname{ind}(C_A) = 1$) such that

$$\text{(2.2)} \qquad A = C_A + N_A, \quad C_A N_A = 0, \quad N_A C_A = 0;$$

see, for instance, [1, pp. 175–177] or [16, p. 246]. This decomposition is called the *core-nilpotent* decomposition of A and is uniquely determined. In the literature, core matrices are also often called *group matrices* or GP matrices.

From Chipman we have the following definition: Two matrices A and B of the same column number, say n, are said to be *complementary* to one another if $\mathbf{R}^n =$

$\mathcal{R}(A^t) \oplus \mathcal{R}(B^t)$. In the literature, complementary matrices have been studied because of their importance in statistics (see, for instance, Chipman [2], Pringle and Rayner [9], Hartung and Werner [3], and Werner [14]). In [15] (see also [14], [13], [17]) the following weaker or stronger versions of that notion are studied.

(a) B is said to be *weakly complementary* to A if $\mathcal{R}(A^t) \cap \mathcal{R}(B^t) = \{0\}$.

(b) B is said to be *(weakly) bicomplementary* to A if B and B^t are (weakly) complementary to A and A^t, respectively.

A pair of weakly bicomplementary matrices is also often said to be a pair of *disjoint* matrices (also written $A+B = A \oplus B$); cf. Mitra [5]. The connections between these concepts and the concept of generalized inversion are discussed in detail in [15] and [5]. Below we cite only those results that are of interest to us in this paper.

THEOREM 2.2 (see [15, p. 369]). *For $A \in \mathbf{R}^{n\times n}$, let $\mathcal{M} \in \mathcal{N}_c(A)$ and let $\mathcal{S} \in \mathcal{R}_c(A)$. Further, let H be a matrix of basis vectors for $\mathcal{M}^\perp$, and let T be a matrix of basis vectors for $\mathcal{S}$. If we define*

$$B = TH^t,$$

then B satisfies $\mathcal{N}(B) = \mathcal{M}$ and $\mathcal{R}(B) = \mathcal{S}$, and so B is bicomplementary to A.

Theorem 2.3 is also well known (cf. [15, pp. 359–364] in combination with [6, p. 240]).

THEOREM 2.3. *For given $A, B \in \mathbf{R}^{m\times n}$, the following conditions are equivalent:*

(i) $A + B = A \oplus B$;

(ii) $\mathcal{R}(A+B) = \mathcal{R}(A) \oplus \mathcal{R}(B)$;

(iii) $\operatorname{rank}(A+B) = \operatorname{rank}(A) + \operatorname{rank}(B)$;

(iv) $\mathcal{N}(A+B) = \mathcal{N}(A) \cap \mathcal{N}(B)$, $\mathcal{M} = [\mathcal{M} \cap \mathcal{N}(A)] \oplus [\mathcal{M} \cap \mathcal{N}(B)]$ *for each* $\mathcal{M} \in \mathcal{N}_c(A+B)$;

(v) $(A+B)\{1\} \subseteq A\{1\}$;

(vi) $(A+B)\{\mathcal{M}, \mathcal{S}\} \subseteq A\{\mathcal{M} \cap \mathcal{N}(B), \mathcal{S} \oplus \mathcal{R}(B)\}$ *for each* $\mathcal{M} \in \mathcal{N}_c(A+B)$ *and* $\mathcal{S} \in \mathcal{R}_c(A+B)$;

(vii) $A \preceq^- A + B$;

(viii) $A^t + B^t = A^t \oplus B^t$.

THEOREM 2.4 (see [15, p. 362]). *If $A + B = A \oplus B$, then*

$$(A+B)^{\{2,\mathcal{M},\mathcal{S}\}} = A^{\{2,\mathcal{M}\cap\mathcal{N}(B),\mathcal{S}\oplus\mathcal{R}(B)\}} + B^{\{2,\mathcal{M}\cap\mathcal{N}(A),\mathcal{S}\oplus\mathcal{R}(A)\}}$$

for every $\mathcal{M} \in \mathcal{N}_c(A+B)$ and every $\mathcal{S} \in \mathcal{R}_c(A+B)$.

3. $\mathcal{G}$-based partial order characterizations. In what follows, let $\mathcal{G}$ be a map on $\mathbf{R}^{m\times n}$ satisfying (1.1) for each $m \times n$ matrix A, and let $\Omega_\mathcal{G}$ be its support. For A, $B \in \Omega_\mathcal{G}$, it is convenient to put

$$\mathcal{G}(A \mid B) := \{GAG \mid G \in \mathcal{G}(B)\}, \tag{3.1}$$

$$\mathcal{G}_r(B) := \{B_r^- \mid B_r^- \in \mathcal{G}(B)\}, \tag{3.2}$$

and

$$\mathcal{G}_r(A \mid B) := \{GAG \mid G \in \mathcal{G}_r(B)\}. \tag{3.3}$$

Observe that $\mathcal{G}$ is semicomplete iff $\mathcal{G}(A \mid A) = \mathcal{G}_r(A)$ for each $A \in \Omega_\mathcal{G}$.

Mitra [6, p. 242] has shown that $\preceq^\mathcal{G}$ as defined in (1.2) need not always correspond to a partial order. It is therefore quite natural to ask for sufficient and/or necessary conditions under which (1.2) defines a partial order. In [6, p. 243], Mitra derived the following sufficient condition for $\preceq^{\mathcal{G}}$ to correspond to a partial order.

THEOREM 3.1. *Let*

$$A \preceq^{\mathcal{G}} B,\ B \text{ not maximal} \implies \mathcal{G}(A \mid B) \subseteq \mathcal{G}(A). \tag{3.4}$$

Then $\preceq^{\mathcal{G}}$ defines a partial order.

Note that a matrix B is called *maximal* relative to $\preceq^{\mathcal{G}}$ if there is no matrix $C \neq B$ such that $B \preceq^{\mathcal{G}} C$. Since $B \preceq^{\mathcal{G}} C$ implies $B \preceq^{-} C$, by Theorem 2.3(iii) $\operatorname{rank}(B) < \min\{m, n\}$ whenever B is not maximal. Further, note that if $B \notin \Omega_{\mathcal{G}}$ and/or B is of full rank (i.e., $\operatorname{rank}(B) = \min\{m, n\}$) then B is maximal relative to $\preceq^{\mathcal{G}}$.

In context with Theorem 3.1 it is further pertinent to mention that the only time when condition (3.4) is invoked by Mitra is in proving that the implication

$$A \preceq^{\mathcal{G}} B,\ B \preceq^{\mathcal{G}} C \implies A \preceq^{\mathcal{G}} C$$

holds true, that is, in proving that the relation $\preceq^{\mathcal{G}}$ is transitive. But there $B \in \Omega_{\mathcal{G}}$ so that, without loss of generality, B can be assumed not to be maximal, for otherwise $B = C$, in which case the implication is trivial. In the same paper, Mitra showed [6, p. 243] that condition (3.4), although sufficient, is in general not necessary for $\preceq^{\mathcal{G}}$ to define a partial order.

In this paper we are especially interested in semicomplete maps. Our next theorem will show, in particular, that if $\mathcal{G}$ is a semicomplete map on the set of square $n \times n$ matrices, then, for $\preceq^{\mathcal{G}}$ to be a partial order, condition (3.4) is not only sufficient but also necessary. The proof follows from [6, Thms. 2.3, 2.4, and 2.5]. In passing, we mention that a different possibility for proving this theorem would be to make use of Theorems 2.2, 2.3, and 2.4 in this paper.

THEOREM 3.2. *Let $\mathcal{G}$ be a semicomplete map on $\mathbf{R}^{n\times n}$. Then the following conditions are equivalent:*

(i) $\preceq^{\mathcal{G}}$ *is a partial order*;
(ii) $A \preceq^{\mathcal{G}} B$, *$B$ not maximal* $\implies \mathcal{G}_r(A \mid B) \subseteq \mathcal{G}_r(A)$;
(iii) $A \preceq^{\mathcal{G}} B$, *$B$ not maximal* $\implies \mathcal{G}(A \mid B) \subseteq \mathcal{G}(A)$.

Next we give Theorem 3.3.

THEOREM 3.3. *Let $\mathcal{G}$ as defined by* (1.1) *be a map on $\mathbf{R}^{m\times n}$ and assume that*

$$\mathcal{G}(A) \neq \emptyset \ \Rightarrow\ \mathcal{G}(A) \cap A\{1,2\} \neq \emptyset. \tag{3.5}$$

Let the relation $\preceq^{\mathcal{G}}$ be defined as in (1.2). *If $A \in \Omega_{\mathcal{G}}$ then A is maximal if and only if* $\operatorname{rank}(A) = \min\{m, n\}$.

Proof. Let $A \in \Omega_{\mathcal{G}}$ and $\operatorname{rank}(A) \neq \min\{m, n\}$. Here we exactly follow the steps in the proof of Theorem 2.5 in [6] to arrive at a matrix $C \neq A$ which dominates A under $\preceq^{\mathcal{G}}$; the modifications required to prove this are obvious. □

Note that a semicomplete map trivially satisfies condition (3.5). Hence Theorem 3.3 holds for a semicomplete map.

That the characterizations in Theorem 3.2 are not necessarily true for a map $\mathcal{G}$ that is defined on the set of nonsquare $m \times n$ matrices (i.e., $m \neq n$) is shown by our next numerical example. Although Example 5 in [6] might also be used for this purpose, the example given below is easier to understand.

Example 3.4. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and the set $\mathcal{A}$ consisting of all those real matrices

$$H_{a,b} := \begin{pmatrix} 1 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ 1 & 0 & a & b \end{pmatrix}$$

for which $a \neq 0$ and/or $b \neq 0$. By checking the corresponding defining equations (G1) and (G2) of (2.1) it is seen that $G \in B\{1,2\}$ and $\mathcal{A} \subseteq A\{1,2\}$. Define the map $\mathcal{G}$ on $\mathbf{R}^{4\times 3}$ by

$$\mathcal{G}(C) = \begin{cases} \mathcal{A} & \text{if } C = A, \\ \{G\} & \text{if } C = B, \\ \emptyset & \text{otherwise.} \end{cases}$$

As is evident, $\mathcal{G}$ is semicomplete. Check that $A \preceq^{\mathcal{G}} B$. Since $\Omega_{\mathcal{G}} = \{A, B\}$, the relation $\preceq^{\mathcal{G}}$ is transitive iff

$$A \preceq^{\mathcal{G}} B,\ B \preceq^{\mathcal{G}} C \Longrightarrow A \preceq^{\mathcal{G}} C$$

holds true. Notice that C satisfies $B \preceq^{\mathcal{G}} C$ iff

$$C = C_{c,d} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & -c \\ d & 0 & -d \end{pmatrix}$$

for some real scalars c and d. Since $A \preceq^{\mathcal{G}} C_{c,d}$ holds irrespective of c and d, it is now clear that the relation $\preceq^{\mathcal{G}}$ defines a partial order. However, observe that $GAG = H_{0,0}$ fails to belong to $\mathcal{A}$. □

In this context it should be mentioned that if the semicomplete map $\mathcal{G}$ is such that for each A its image $\mathcal{G}(A)$ is a singleton or empty then, even in the nonsquare case (i.e., $m \neq n$), condition (3.4) turns out to be necessary and sufficient for $\preceq^{\mathcal{G}}$ to define a partial order. The following result is only slightly more general than Theorem 2.6 in [6] insofar as it allows $\mathcal{G}(A)$ to be empty; however, observe that the proof given in [6, p. 250] actually includes our result.

THEOREM 3.5. *Let $\mathcal{G}$ as defined by* (1.1) *be a semicomplete map with* card $\mathcal{G}(A) \leq 1$ *for each $A \in \mathbf{R}^{m\times n}$. Then for $\preceq^{\mathcal{G}}$ to correspond to a partial order, it is necessary and sufficient that*

$$A \preceq^{\mathcal{G}} B,\ B \text{ not maximal} \Longrightarrow \mathcal{G}(A \mid B) \subseteq \mathcal{G}(A). \tag{3.4}$$

4. $\mathcal{G}$-based partial order extensions. We begin this section with the following result.

THEOREM 4.1. *Let $\preceq^{\mathcal{G}}$ as defined by* (1.2) *correspond to a partial order, and let $\Omega_{\mathcal{G}}$ be a proper subset of $\mathbf{R}^{m\times n}$. The relation $\preceq^{\mathcal{G}}$ can then be extended to a $\mathcal{G}$-based partial order $\preceq^{\mathcal{G}^*}$ that supports the whole of $\mathbf{R}^{m\times n}$.*

Proof. Define the map $\mathcal{G}^*$ by

$$\mathcal{G}^*(A) = \begin{cases} \mathcal{G}(A) & \text{if } A \in \Omega_{\mathcal{G}}, \\ \{A^-_{\max}\} & \text{otherwise,} \end{cases} \tag{4.1}$$

where $A^-_{\max}$ is an arbitrary but fixed full-rank g-inverse of A. Theorem 2.1(iv) tells us that such a g-inverse of A with rank $r := \min\{m, n\}$ does always exist. We next prove that each matrix $A \notin \Omega_{\mathcal{G}}$ is maximal relative to $\preceq^{\mathcal{G}^*}$. To that end let $A \notin \Omega_{\mathcal{G}}$.

Moreover, let B be any matrix such that $A \preceq^{\mathcal{G}^*} B$. Clearly, $A \preceq^{\mathcal{G}^*} B$ iff $(B-A)A^-_{\max} = 0$ and $A^-_{\max}(B-A) = 0$. Since $A^-_{\max}$ is a full-rank matrix, this, of course, happens iff $B - A = 0$. So we arrive at $B = A$, and it is now clear that A is maximal relative to $\preceq^{\mathcal{G}^*}$. With this observation in mind, it is evident that $\preceq^{\mathcal{G}^*}$ inherits the transitivity property from $\preceq^{\mathcal{G}}$. As $\preceq^{\mathcal{G}^*}$ supports each matrix, the proof is complete. □

The proof of the preceding theorem has shown that there are many trivial ways to extend a $\mathcal{G}$-based partial order on the whole of $\mathbf{R}^{m\times n}$. We call an extension $\preceq^{\mathcal{G}^*}$ a *trivial extension* of $\preceq^{\mathcal{G}}$ if

$$A \preceq^{\mathcal{G}^*} B \iff A \preceq^{\mathcal{G}} B$$

for all pairs of matrices with $A \neq B$. In fact, this means, in an obvious sense, that $\preceq^{\mathcal{G}}$ and $\preceq^{\mathcal{G}^*}$ are *equivalent* relations.

Now it seems quite natural to ask the following: Given a $\mathcal{G}$-based partial order that excludes from its support a chunk of $\mathbf{R}^{m\times n}$, does it always admit a *nontrivial* $\mathcal{G}$-based partial order extension that supports the whole of $\mathbf{R}^{m\times n}$?

Concerning the sharp order $\preceq^{\#}$, Mitra [7, Thm. 2.1] has already given an affirmative answer to this question. In order to discuss that result we need some further notation. Recall that if A is a square $n \times n$ matrix, then

$$A = C_A + N_A$$

stands for the uniquely determined core-nilpotent decomposition of A; see (2.2) in §2. Let us write $A \preceq^{\dagger} B$ if

$$A \preceq^{-} B \quad \text{and} \quad C_A \preceq^{\#} C_B.$$

Note that the relation $\preceq^{\dagger}$ is an *extension* of the sharp order $\preceq^{\#}$ in the sense that

$$A \preceq^{\#} B \implies A \preceq^{\dagger} B.$$

Since the minus order and the sharp order are partial orders, $\preceq^{\dagger}$ is also a partial order. Moreover, observe that by Theorem 2.1 in [7] the relation $\preceq^{\dagger}$ is equivalent to the $\mathcal{G}$-based relation $\preceq^{\mathcal{G}^0}$, where the map $\mathcal{G}^0$ is defined by

$$(4.2) \qquad \mathcal{G}^0(A) := \begin{cases} \{A^{\#}\} & \text{if } \operatorname{ind}(A) = 1, \\ \{A^- \mid \mathcal{R}(C_A) \subseteq \mathcal{R}(A^-),\ \mathcal{N}(A^-) \subseteq \mathcal{N}(C_A)\} & \text{otherwise.} \end{cases}$$

It is pertinent to prove here that this map $\mathcal{G}^0$ fails to be semicomplete whenever $n \geq 3$. For this purpose, let $n \geq 3$ and let $A \in \mathbf{R}^{n\times n}$ be a matrix that is neither core nor nilpotent. Then $A = C_A + N_A$, $C_A \neq 0$, $N_A \neq 0$, $1 < \operatorname{rank}(A) = \operatorname{rank}(C_A) + \operatorname{rank}(N_A) < n$. So it is possible to choose an $\mathcal{M} \in \mathcal{N}_c(A)$ such that $\mathcal{R}(C_A) \not\subseteq \mathcal{M}$ (see §2). In addition, choose $\mathcal{S} \in \mathcal{R}_c(A)$. From Theorem 2.2 we then know that there exists a matrix B such that $\mathcal{R}(B) = \mathcal{S}$, $\mathcal{N}(B) = \mathcal{M}$, and B is bicomplementary to A. Note that the matrix $A + B$ is hence, in particular, nonsingular. From Theorem 2.3 along with Theorems 2.4 and 2.1 we get

$$(4.3) \qquad (A+B)^{-1}A(A+B)^{-1} = A^{\{2,\mathcal{N}(B),\mathcal{R}(B)\}}.$$

But although trivially $(A+B)^{-1} \in \mathcal{G}^0(A)$, $(A+B)^{-1}A(A+B)^{-1}$ does not belong to $\mathcal{G}^0(A)$ because $\mathcal{R}(C_A) \not\subseteq \mathcal{N}(B) = \mathcal{M}$. But now it is clear that $\mathcal{G}^0$ is not a semicomplete map if $n \geq 3$. The only exceptions are the cases $n = 1$ and $n = 2$. Every matrix of order 1×1 is a core matrix. Here the map is clearly semicomplete. A matrix A of order 2×2 is either a core matrix or is nilpotent. If A is nilpotent then $\mathcal{G}^0(A) = A\{1\}$, so that the map $\mathcal{G}^0$ is also trivially seen to be semicomplete in that case.

For A, let P_A denote the well-defined projector onto $\mathcal{R}(C_A)$ along $\mathcal{N}(C_A)$; that is, let $P_A = C_A C_A^{\#}$. Since

$$C_A^{\#} + (I - P_A)A^-(I - P_A) \in \mathcal{G}^0(A) \quad \text{if} \quad \mathrm{ind}(A) > 1$$

(see [7, Lem. 2.3]), clearly $\mathcal{G}^0(A) \neq \emptyset$ for each A. Combining observations now shows that $\preceq^{\mathcal{G}^0}$ is indeed a $\mathcal{G}$-based partial order extension of the sharp order that supports the whole of $\mathbf{R}^{n\times n}$. This is shown in Mitra [7].

The theorem that follows is somewhat different; it gives a $\mathcal{G}$-based extension $\preceq^{\mathcal{G}_*}$ of $\preceq^{\#}$ and various equivalent descriptions of the underlying map $\mathcal{G}_*$.

THEOREM 4.2. *For square $n \times n$ matrices, let $\mathcal{G}^0$ and $\mathcal{G}_*$ be defined by* (4.2) *and*

$$\mathcal{G}_*(A) := \begin{cases} \{A^{\#}\} & \text{if } \mathrm{ind}(A) = 1, \\ \{A^- \mid A^- C_A A^- = C_A^{\#}\} & \text{otherwise,} \end{cases} \tag{4.4}$$

respectively. Then we have the following.

(i) *For noncore square matrices A, $\mathcal{G}_*(A)$ allows the following equivalent descriptions:*

(4.5) $\mathcal{G}_*(A) = \{A^- \mid A^- C_A = C_A A^-\}$,

(4.6) $\mathcal{G}_*(A) = \{A^- \mid \mathcal{R}(C_A) \subseteq \mathcal{R}(A^-A),\ \mathcal{N}(AA^-) \subseteq \mathcal{N}(C_A)\}$,

(4.7) $\mathcal{G}_*(A) = \{C_A^{\#} + (I - P_A)Z(I - P_A) \mid Z \in A\{1\}\}$, *where* $P_A = C_A C_A^{\#}$.

(ii) *The relation $\preceq^{\mathcal{G}_*}$ is a semicomplete partial order extension of the sharp order $\preceq^{\#}$ and supports the whole of $\mathbf{R}^{n\times n}$. Moreover, if $n \geq 3$ then $\mathcal{G}_*$ is properly finer than $\mathcal{G}^0$. Precisely, $\mathcal{G}_*(A) \subseteq \mathcal{G}^0(A)$ for each matrix A, and $\mathcal{G}_*(A) \neq \mathcal{G}^0(A)$ whenever A is neither core nor nilpotent.*

Proof. (i): Let A be a noncore square matrix. It is easily checked that (4.7) is a subset of $A\{1\}$. We now show that $A^- \in$ the set (4.7) $\Rightarrow A^- \in$ the set (4.4) $\Rightarrow A^- \in$ the set (4.5) $\Rightarrow A^- \in$ the set (4.6) $\Rightarrow A^- \in$ the set (4.7), thus establishing equivalence. Let $A^- = C_A^{\#} + (I - P_A)Z(I - P_A)$. Then $A^- C_A A^- = [C_A^{\#} + (I - P_A)Z(I - P_A)]C_A C_A^{\#} C_A [C_A^{\#} + (I - P_A)Z(I - P_A)] = C_A^{\#} C_A C_A^{\#} C_A C_A^{\#} = C_A^{\#}$. This implies $A^- C_A = A^- C_A A^- C_A = C_A^{\#} C_A = C_A C_A^{\#} = C_A A^- C_A A^- = C_A A^-$ using Theorem 2.3(v) since

$$A = C_A \oplus N_A,$$

which in turn shows that

$$C_A = C_A A^- C_A = A^- C_A^2 = A^- A C_A.$$

This is equivalent to $\mathcal{R}(C_A) \subseteq \mathcal{R}(A^-A)$. Similarly,

$$C_A = C_A A^- C_A = C_A^2 A^- = C_A A A^-$$

or, equivalently, $\mathcal{N}(AA^-) \subseteq \mathcal{N}(C_A)$. From the pair of equivalences just established it is seen that $A^- \in$ the set (4.6) implies that the matrix $X = A^-$ satisfies the simultaneous system of equations

$$XC_A^2 = C_A = C_A^2 X. \tag{4.8}$$

The matrix $X = C_A^{\#}$ is a particular solution of (4.8). Using Lemma 2.3.1 in [10], we thus have $X = C_A^{\#} + (I - P_A)Z(I - P_A)$, Z arbitrary as the expression for the general solution of (4.8). But then it is immediate that in order that $X \in A\{1\}$ we must have $Z \in N_A\{1\}$. Since $(I - P_A)X(I - P_A) = (I - P_A)Z(I - P_A)$, it is now clear that (4.6) implies (4.7).

(ii): From part (i) it is clear that $\preceq^{\mathcal{G}_*}$ is a $\mathcal{G}$-based extension of $\preceq^{\#}$ that supports the whole of $\mathbf{R}^{n \times n}$. Semicompleteness of $\mathcal{G}_*$ follows from (4.6) by observing that $\mathcal{R}(A^-AA^-A) = \mathcal{R}(A^-A)$ and $\mathcal{N}(AA^-AA^-) = \mathcal{N}(AA^-)$. Next note that the proof of Theorem 2.1 in [7] can be used word for word to establish that $\preceq^{\dagger}$ is equivalent to $\preceq^{\mathcal{G}_*}$. The relation $\preceq^{\mathcal{G}_*}$ therefore corresponds to a $\mathcal{G}$-based partial order extension of the sharp order and is equivalent to the $\mathcal{G}$-based relation $\preceq^{\mathcal{G}^0}$. Comparing (4.2) with (4.6) shows that $\mathcal{G}_*(A) \subseteq \mathcal{G}^0(A)$ holds for each matrix A. That $\mathcal{G}_*$ is properly finer than $\mathcal{G}^0$ whenever $n \geq 3$ is an easy consequence of the lines directly following (4.2). To see this, let A be neither core nor nilpotent and consider the matrix $A + B$ constructed there. As seen above, $(A+B)^{-1} \in \mathcal{G}^0(A)$. That $(A+B)^{-1}$ fails to belong to $\mathcal{G}^*(A)$ follows from (4.3) since $\mathcal{R}(C_A) \not\subseteq \mathcal{N}(B)$. Note that $\mathcal{G}_*(A) = \mathcal{G}^0(A)$ if A is core or nilpotent. This completes the proof. □

Notice that $\mathcal{G}_*$ as defined by (4.4) is semicomplete and that $\operatorname{card}\mathcal{G}_*(A) > 1$ whenever $\operatorname{ind}(A) > 1$. It is therefore interesting to mention that the following example will show that for $n \geq 3$ there does not exist any $\mathcal{G}$-based semicomplete partial order extension $\preceq^{\tilde{\mathcal{G}}}$ of the sharp order $\preceq^{\#}$ such that $\tilde{\mathcal{G}}(A)$ is a singleton set for all A.

Example 4.3. First, let $n = 3$. Consider the nilpotent matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By virtue of Theorem 3.2 it is sufficient to show that for each reflexive g-inverse A_r^- of A we can always find a matrix B of rank 2 and index 1 such that $B^{\#}AB^{\#} \neq A_r^-$, although $AA_r^- = BA_r^-$ and $A_r^-A = A_r^-B$.

Observe that

$$A\{1,2\} = \left\{ \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} (a \quad b \quad 1) \mid a, b, c, d \in \mathbf{R} \right\}$$

is an efficient parametrization of the set of all reflexive g-inverses of the nilpotent matrix A. This can be easily seen, for instance, by means of (G1) and (G2) in (2.1). Consider an arbitrary but fixed reflexive g-inverse

$$A_r^- = \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} (a \quad b \quad 1)$$

of A and put

$$x := \begin{pmatrix} 1 \\ f \\ -(a + fb) \end{pmatrix} \quad \text{and} \quad y^t := \left(\frac{-fd - ca}{f - c} \quad \frac{d + a + fb}{f - c} \quad 1 \right),$$

where $f := 1 - cc^{\dagger}$. Set

$$B := A + xy^t.$$

Since

$$f = 0 \iff c \neq 0 \quad \text{and} \quad f = 1 \iff c = 0, \tag{4.9}$$

$fc = 0$ and $f^2 = f$. This in turn implies $y^t x = 0$. As $A_r^- x = 0$ and $y^t A_r^- = 0$, clearly $A_r^- A = A_r^- B$ and $AA_r^- = BA_r^-$. Next notice that $B^3 = Axy^tA + xy^tAxy^t = A + xy^t = B$. Therefore, $\operatorname{ind}(B) = 1$ and $B^\# = B$ (note §2). Recalling (4.9), the desired result now follows from

$$\begin{aligned} B^\# AB^\# &= BAB \\ &= xy^t \\ &= \begin{pmatrix} 1 \\ \mathbf{f} \\ -(c+fb) \end{pmatrix} \begin{pmatrix} \dfrac{-fd-ca}{f-c} & \dfrac{d+a+fb}{f-c} & 1 \end{pmatrix} \\ &\neq \begin{pmatrix} 1 \\ \mathbf{c} \\ d \end{pmatrix} \begin{pmatrix} a & b & 1 \end{pmatrix} \\ &= A_r^- . \end{aligned}$$

If $n > 3$, then the proof follows along similar lines and is thus omitted. □

5. Semicomplete $\mathcal{G}$-based partial order extensions. In this final section, let $\mathcal{G}$ as defined by (1.1) be a semicomplete map on $\mathbf{R}^{m\times n}$ and let this map induce a partial order $\preceq^{\mathcal{G}}$.

Recall that each *full-rank* matrix A (i.e., $\operatorname{rank}(A) = \min\{m,n\}$) is maximal relative to $\preceq^{\mathcal{G}}$. It is pertinent to mention that modifying $\mathcal{G}(\cdot)$ for one or more full-rank matrices will lead to a new map but that this modified map continues to induce exactly the same relation $\preceq^{\mathcal{G}}$ as $\mathcal{G}$ (at least concerning all pairs (A,B) of matrices with $A \neq B$). Without loss of generality, let us henceforth assume that $\mathcal{G}(A) = A\{1\}$ for each full-rank matrix A.

Let us further assume that $\Omega_{\mathcal{G}} \neq \mathbf{R}^{m\times n}$. Then $A \notin \Omega_{\mathcal{G}}$ for some matrix A with $\operatorname{rank}(A) < \min\{m,n\}$. For each matrix $B \in \mathbf{R}^{m\times n}$, let $\mathcal{UP}_{\mathcal{G}}(B) := \{A \mid A \preceq^{\mathcal{G}} B\}$ denote the set of all those matrices that are *upstream* of B. Moreover, whenever $B \notin \Omega_{\mathcal{G}}$ is such that $\mathcal{UP}_{\mathcal{G}}(B) \neq \emptyset$, put

$$\mathcal{G}_0(B) := \{B^- \mid B^-AB^- \in \mathcal{G}(A) \text{ for each } A \in \mathcal{UP}_{\mathcal{G}}(B)\}. \tag{5.1}$$

Finally, define a new map $\mathcal{G}^*$ (in respect to $\mathcal{G}$) by

$$\mathcal{G}^*(B) := \begin{cases} \mathcal{G}(B) & \text{if } B \in \Omega_{\mathcal{G}}; \\ \mathcal{G}_0(B) & \text{if } B \notin \Omega_{\mathcal{G}},\ \mathcal{UP}_{\mathcal{G}}(B) \neq \emptyset; \\ B\{1\} & \text{otherwise.} \end{cases} \tag{5.2}$$

For square matrices (i.e., $m = n$), Theorem 3.2 could now tend to make one believe that $\mathcal{G}^*$ as defined by (5.2) does induce, in any case, a (nontrivial) semicomplete partial order extension $\preceq^{\mathcal{G}^*}$ of $\preceq^{\mathcal{G}}$ which, unlike $\preceq^{\mathcal{G}}$, supports the whole of $\mathbf{R}^{n\times n}$. That this, however, is erroneous is seen by the following example.

Example 5.1. Consider the matrices

$$A_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By checking the corresponding defining equations (G1) and (G2) of (2.1) it is seen that

$$G_1 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad G_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

are reflexive g-inverses of A_1 and A_2, respectively. Define the map

$$\mathcal{G}: \quad \mathbf{R}^{4\times 4} \to \mathcal{P}(\mathbf{R}^{4\times 4})$$

by

$$\mathcal{G}(A) = \begin{cases} \{G_1\} & \text{if } A = A_1, \\ \{G_2\} & \text{if } A = A_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is obvious that $\mathcal{G}$ is semicomplete and that the induced order $\preceq^{\mathcal{G}}$ defines a partial order with support $\Omega_{\mathcal{G}} = \{A_1, A_2\}$. Now let $\tilde{\mathcal{G}}$ be a semicomplete extension of $\mathcal{G}$ such that the induced order $\preceq^{\tilde{\mathcal{G}}}$ corresponds to a partial order.

Suppose that $\preceq^{\tilde{\mathcal{G}}}$ supports $\mathbf{R}^{4\times 4}$; that is, $\Omega_{\tilde{\mathcal{G}}} = \mathbf{R}^{4\times 4}$. Then $B \in \Omega_{\tilde{\mathcal{G}}}$ or, equivalently, $\tilde{\mathcal{G}}_r(B) \neq \emptyset$. On the one hand, by Theorem 3.3, B cannot be maximal with respect to $\preceq^{\tilde{\mathcal{G}}}$ because B is singular. Hence, by Theorem 3.2, $\tilde{\mathcal{G}}_r(A \mid B) \subseteq \tilde{\mathcal{G}}_r(A)$ for each $A \preceq^{\tilde{\mathcal{G}}} B$. For $i = 1, 2$, clearly $\tilde{\mathcal{G}}_r(A_i) = \mathcal{G}_r(A_i) = \{G_i\}$. Since $BG_1 = A_1G_1$ and $G_1B = G_1A_1$, $A_1 \preceq^{\tilde{\mathcal{G}}} B$. Likewise, it is seen that $A_2 \preceq^{\tilde{\mathcal{G}}} B$. Consequently,

$$B_r^- A_i B_r^- = G_i \text{ for } i = 1, 2 \text{ and for each } B_r^- \in \tilde{\mathcal{G}}_r(B).$$

On the other hand, recall that $\tilde{\mathcal{G}}_r(B) \subseteq B\{1, 2\}$. It is easy to check that

$$B\{1\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ y_1 & y_2 & y_3 & z \end{pmatrix} \mid x_i, y_i, z \in \mathbf{R} \quad (i = 1, 2, 3) \right\}.$$

Therefore,

$$B\{1, 2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{pmatrix} \mid x_i, y_i \in \mathbf{R} \quad (i = 1, 2, 3) \right\}$$

because $B\{1, 2\} = B\{1\}\{B\}B\{1\}$ (recall Theorem 2.1). From this we get

$$(5.3) \quad \{B_r^- A_1 B_r^- \mid B_r^- \in B\{1, 2\}\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \end{pmatrix} \mid x_i, y_i \in \mathbf{R} \quad (i = 1, 2) \right\}$$

and

$$(5.4)\quad \{B_r^- A_2 B_r^- \mid B_r^- \in B\{1,2\}\} = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{pmatrix} \mid x_i, y_i \in \mathbf{R} \quad (i = 2, 3) \right\}.$$

Invoking (5.3) and (5.4), respectively, now yields

$$B_r^- A_1 B_r^- = G_1 \text{ iff } y_1 = 1, y_2 = 0, x_1 = 1, x_2 = 1$$

and

$$B_r^- A_2 B_r^- = G_2 \text{ iff } y_2 = 1, y_3 = -1, x_2 = 0, x_3 = 0.$$

Since it is impossible to simultaneously set y_2 equal to 0 and equal to 1, there does not exist any reflexive g-inverse B_r^- of B simultaneously satisfying

$$B_r^- A_1 B_r^- = G_1 \quad \text{and} \quad B_r^- A_2 B_r^- = G_2.$$

But, in fact, $\tilde{\mathcal{G}}_r(B) \subseteq B\{1,2\}$. Consequently, $\tilde{\mathcal{G}}_r(B) = \emptyset$, which is a contradiction to $\Omega_{\tilde{\mathcal{G}}} = \mathbf{R}^{4\times 4}$. As desired, the relation $\preceq^{\mathcal{G}}$ thus fails to possess a $\mathcal{G}$-based semicomplete partial order extension $\preceq^{\tilde{\mathcal{G}}}$ that supports the whole of $\mathbf{R}^{4\times 4}$. □

Example 5.1 thus exhibits that we cannot always expect $\mathcal{G}^*$ to support the whole set of $m \times n$ matrices. Nevertheless, we will next show that $\mathcal{G}^*$ is semicomplete whenever $\mathcal{G}$ is semicomplete.

THEOREM 5.2. *Let $\mathcal{G}$ as defined by* (1.1) *be a semicomplete map, and let $\mathcal{G}^*$ be defined according to* (5.2). *Then $\mathcal{G}^*$ is also semicomplete.*

Proof. Let $B \in \Omega_{\mathcal{G}^*}$. Then $\mathcal{G}^*(B) \neq \emptyset$. We must show that $B^-BB^- \in \mathcal{G}^*(B)$ whenever $B^- \in \mathcal{G}^*(B)$. If $B \in \Omega_{\mathcal{G}}$, then $\mathcal{G}^*(B) = \mathcal{G}(B)$, and the desired result follows from the semicompleteness of $\mathcal{G}$. Next, assume that $B \notin \Omega_{\mathcal{G}}$. Then either $\mathcal{UP}_{\mathcal{G}}(B) = \emptyset$ or $\mathcal{UP}_{\mathcal{G}}(B) \neq \emptyset$. In the former case $\mathcal{G}^*(B) = B\{1\}$, so that $B^-BB^- \in \mathcal{G}^*(B)$ holds trivially for each $B^- \in \mathcal{G}^*(B)$ (note Theorem 2.1(ii)). In the latter case $\mathcal{G}^*(B) = \mathcal{G}_0(B)$. But then, by the definition of $\mathcal{G}_0(B)$, $B^- \in \mathcal{G}^*(B)$ iff $B^-AB^- \in \mathcal{G}(A)$ for each $A \in \mathcal{UP}_{\mathcal{G}}(B)$. Since $A \in \mathcal{UP}_{\mathcal{G}}(B)$ iff $A \preceq^{\mathcal{G}} B$, then $A \preceq^- B$ and so, in view of Theorem 2.3(ii) and (v), $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ or, equivalently, $BB^-A = A = AB^-B$. Consequently, $B^-BB^-AB^-BB^- = B^-AB^- \in \mathcal{G}(A)$. Since $B_r^- := B^-BB^-$ is a reflexive g-inverse of B and $B_r^-AB_r^- \in \mathcal{G}(A)$ is satisfied, the proof is done. □

For the $\mathcal{G}$-based sharp order $\preceq^{\#}$ (set $\mathcal{G}(A) = \{A^{\#}\}$ if $\text{ind}(A) = 1$ and $\mathcal{G}(A) = \emptyset$ if $\text{ind}(A) > 1$), Theorem 5.5 will even show that the associated relation $\preceq^{\mathcal{G}^*}$ corresponds to a partial order. In order to prove this, we need the following result.

THEOREM 5.3. *For square matrices, let the semicomplete $\mathcal{G}$-based relations $\preceq^{\mathcal{G}}$ and $\preceq^{\tilde{\mathcal{G}}}$ both correspond to partial orders. Further, let $\mathcal{G}^*$ be defined in respect to $\mathcal{G}$ according to* (5.2). *If $\preceq^{\tilde{\mathcal{G}}}$ is a $\mathcal{G}$-based extension of $\preceq^{\mathcal{G}}$, then*

$$(5.5)\qquad \tilde{\mathcal{G}}(A) \subseteq \mathcal{G}^*(A)$$

for each A. Thus $\preceq^{\tilde{\mathcal{G}}}$ is finer than $\preceq^{\mathcal{G}^*}$; that is,

$$(5.6)\qquad A \preceq^{\tilde{\mathcal{G}}} B \implies A \preceq^{\mathcal{G}^*} B.$$

Note that (5.5) *and* (5.6) *are not equivalent conditions. In general* (5.5) $\Rightarrow$ (5.6). *The reverse implication, however, need not always be true.*

Proof. By the definition of a $\mathcal{G}$-based extension, clearly $\tilde{\mathcal{G}}(B) = \mathcal{G}(B)$ whenever $\mathcal{G}(B) \neq \emptyset$. Since in that case also $\mathcal{G}^*(B) = \mathcal{G}(B)$, we trivially arrive at $\tilde{\mathcal{G}}(B) \subseteq \mathcal{G}^*(B)$.

Let us next consider the case when $\tilde{\mathcal{G}}(B) \neq \emptyset$ but $\mathcal{G}(B) = \emptyset$. Note that, by Theorem 3.3, B is maximal relative to $\preceq^{\tilde{\mathcal{G}}}$ iff B is nonsingular. First, let B be not maximal and suppose that there does exist some proper predecessor of B relative to $\preceq^{\tilde{\mathcal{G}}}$, say A. Then $A \neq B$ and $A \preceq^{\tilde{\mathcal{G}}} B$. Since $\tilde{\mathcal{G}}$ is semicomplete and $\preceq^{\tilde{\mathcal{G}}}$ corresponds to a partial order, we know from Theorem 3.2 that $\tilde{\mathcal{G}}(A \mid B) \subseteq \tilde{\mathcal{G}}(A)$ for each $A \preceq^{\tilde{\mathcal{G}}} B$. Therefore, in particular, $\tilde{\mathcal{G}}(A \mid B) \subseteq \mathcal{G}(A)$ for each A with $A \preceq^{\mathcal{G}} B$, thus showing that $\tilde{\mathcal{G}}(B) \subseteq \mathcal{G}_0(B) = \mathcal{G}^*(B)$ when $\mathcal{UP}_{\mathcal{G}}(B) \neq \emptyset$. When $\mathcal{UP}_{\mathcal{G}}(B) = \emptyset$, then trivially $\tilde{\mathcal{G}}(B) \subseteq B\{1\} = \mathcal{G}^*(B)$. Second, let B be not maximal and suppose $\mathcal{UP}_{\tilde{\mathcal{G}}}(B) = \{B\}$. Then $\mathcal{UP}_{\mathcal{G}}(B) = \emptyset$, so that $\mathcal{G}^*(B) = B\{1\}$. Since $\tilde{\mathcal{G}}(B) \subseteq B\{1\}$, again $\tilde{\mathcal{G}}(B) \subseteq \mathcal{G}^*(B)$. Third, let B be maximal relative to $\preceq^{\tilde{\mathcal{G}}}$. Since $\tilde{\mathcal{G}}(B) \neq \emptyset$, B is nonsingular. Consequently, $\tilde{\mathcal{G}}(B) = \{B^{-1}\}$. Recall that in the beginning of this section we saw that modifying $\mathcal{G}(C)$ in case of a nonsingular matrix C has no effect on the induced relation; to avoid unnecessary considerations we thus agreed to assume that $\mathcal{G}(C) = \{C^{-1}\}$ for each nonsingular matrix C. Hence $\mathcal{G}^*(B) = \{B^{-1}\}$, so that trivially $\tilde{\mathcal{G}}(B) = \mathcal{G}^*(B)$.

To complete the proof we finally must consider the case when $\tilde{\mathcal{G}}(B) = \emptyset$. Needless to say, the inclusion $\tilde{\mathcal{G}}(B) \subseteq \mathcal{G}^*(B)$ holds trivially in this case. □

This theorem admits the following interesting corollary; its proof is straightforward and thus is omitted. Observe that Example 5.1 is in accordance with this corollary.

COROLLARY 5.4. *For square $n \times n$ matrices, let $\preceq^{\mathcal{G}}$ denote a $\mathcal{G}$-based semicomplete partial order excluding from its support at least one singular matrix. Let $\mathcal{G}^*$ be defined by* (5.2). *For a $\mathcal{G}$-based semicomplete partial order extension of $\preceq^{\mathcal{G}}$ which supports all $n \times n$ matrices to exist it is then necessary that $\mathcal{G}^*(A) \neq \emptyset$ for each singular matrix A.*

Theorem 5.3 now enables us to prove the following preannounced result regarding the sharp order.

THEOREM 5.5. *Let $\mathcal{G}^*$ be defined according to* (5.2) *in respect to the usual semicomplete map $\mathcal{G}$ which belongs to the $\mathcal{G}$-based sharp order $\preceq^{\#}$. Then $\mathcal{G}^*$ induces a semicomplete partial order extension of $\preceq^{\#}$ that supports the set of all square matrices. Moreover, $\mathcal{G}^*$ is equal to the map $\mathcal{G}_*$ which was introduced in Theorem* 4.2.

Proof. Let $\mathcal{G}_*$ denote the map introduced in Theorem 4.2 by (4.4). Let A be any square matrix, and let $A = C_A + N_A$ again be its core-nilpotent decomposition. Then $C_A \preceq^{\#} A$ since $C_A^{\#} N_A = N_A C_A^{\#} = 0$. Consequently, $C_A \in \mathcal{UP}_{\mathcal{G}}(A)$, which in turn implies $\mathcal{G}^*(A) \subseteq \mathcal{G}_*(A)$. To prove the converse inclusion, note that, by Theorem 4.2, $\preceq^{\mathcal{G}_*}$ is semicomplete. Hence, in view of Theorem 5.3, $\mathcal{G}_*(A) \subseteq \mathcal{G}^*(A)$. Combining observations results in $\mathcal{G}^*(A) = \mathcal{G}_*(A)$, and our claims follow from Theorem 4.2. □

In context with Theorems 5.5, 5.3, and 4.2 it is pertinent to mention the following. Consider the maps $\mathcal{G}^0$ and $\mathcal{G}_*$ defined by (4.2) and (4.4), respectively. From Theorem 4.2 it is known that if $n \geq 3$ then $\mathcal{G}_*$ is properly finer than $\mathcal{G}^0$. Theorem 5.5 now tells us that $\mathcal{G}_* = \mathcal{G}^*$. Since $\preceq^{\mathcal{G}^0}$ is equivalent to the partial order $\preceq^{\dagger}$ (see §4), this seems to contradict Theorem 5.3. Fortunately, however, from the lines directly following (4.2) we already know that $\mathcal{G}^0$ fails to be a semicomplete map whenever $n \geq 3$.

For what follows it is convenient to call $\mathcal{G}$ as defined by (1.1) a *property-p* map if $\mathcal{G}$ is semicomplete and condition (3.4) of §3 is satisfied. If $\mathcal{G}$ is a property-p map, the induced relation $\preceq^{\mathcal{G}}$ as defined by (1.2) is called a *property-p* relation. Since a name should give some aid in visualizing the notion, it is natural to ask the following: Where does the name *"property-p"* come from? The answer is (at least implicitly) already given by Theorem 3.1. For notice that, according to this theorem, each property-p relation is a partial order. In other words, to possess this property "p" is a sufficient condition for relation $\preceq^{\mathcal{G}}$ to define a partial order. In the case of square matrices (i.e., when $m = n$), we even know from Theorem 3.2 that a semicomplete $\mathcal{G}$-based relation $\preceq^{\mathcal{G}}$ corresponds to a partial order iff it is a property-p relation.

Motivated by Theorem 5.5 one might conjecture that, even in the nonsquare case (i.e., when $m \neq n$), the relation $\preceq^{\mathcal{G}^*}$ does always correspond to a partial order, provided $\mathcal{G}$ is a property-p map. The next part of this paper is devoted to establishing that this is indeed the case. For that purpose we need the following lemma.

LEMMA 5.6. *Let $\mathcal{G}$ be a property-p map, and let $\mathcal{G}^*$ be defined in respect to $\mathcal{G}$ according to* (5.2). *If $A \preceq^{\mathcal{G}} B$ and $B \preceq^{\mathcal{G}^*} C$, then $A \preceq^{\mathcal{G}} C$.*

Proof. If $A = B$ and/or $B = C$, then trivially $A \preceq^{\mathcal{G}} C$. Henceforth, let $A \neq B$ and let $B \neq C$. Then $A \in \mathcal{UP}_{\mathcal{G}}(B)$ and $\mathcal{G}^*(B) \neq \emptyset$. Since $\mathcal{G}$ is a property-p map, by Theorem 3.1 clearly $A \preceq^{\mathcal{G}} C$ whenever $B \in \Omega_{\mathcal{G}}$. Now let $B \notin \Omega_{\mathcal{G}}$. Then $\mathcal{G}^*(B) = \mathcal{G}_0(B)$. Since $B \preceq^{\mathcal{G}^*} C$, $(C - B)B^- = 0$ and $B^-(C - B) = 0$ for some suitable $B^- \in \mathcal{G}_0(B)$. This in turn implies

$$(5.7a)\qquad (C - B)B^-AB^- = 0,$$

$$(5.7b)\qquad B^-AB^-(C - B) = 0.$$

By the definition of $\mathcal{G}_0(B)$, $B^-AB^- \in \mathcal{G}(A)$. That $B\{1\} \subseteq A\{1\}$, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ follows from $A \preceq^{\mathcal{G}} B$ by means of Theorem 2.3. But then $BB^-AB^- = AB^-$ because BB^- is a projector onto $\mathcal{R}(B)$. Moreover, $AB^-AB^- = AB^-$. So $BB^-AB^- = AB^-AB^-$ or, equivalently,

$$(5.8a)\qquad (B - A)B^-AB^- = 0.$$

Since B^-B is a projector along $\mathcal{N}(B)$, we likewise get $B^-AB^-B = B^-AB^-A$ or, equivalently,

$$(5.8b)\qquad B^-AB^-(B - A) = 0.$$

Combining (5.7) with (5.8) yields

$$(C - A)B^-AB^- = 0, \qquad B^-AB^-(C - A) = 0.$$

Since $B^-AB^- \in \mathcal{G}(A)$, $A \preceq^{\mathcal{G}} C$, as claimed. □

THEOREM 5.7. *Let $\mathcal{G}$ be a property-p map, and let $\mathcal{G}^*$ be defined in respect to $\mathcal{G}$ according to* (5.2). *Then $\mathcal{G}^*$ is also a property-p map. Although the relation $\preceq^{\mathcal{G}^*}$ thus defines a partial order it need not necessarily support each matrix.*

Proof. That $\preceq^{\mathcal{G}^*}$ does not necessarily support each matrix follows from Example 5.1. That $\mathcal{G}^*$ is semicomplete is the result of Theorem 5.2. In view of Theorem 3.1 we therefore only have to prove that the implication

$$A \preceq^{\mathcal{G}^*} B,\ B \text{ not maximal} \Longrightarrow \mathcal{G}^*(A \mid B) \subseteq \mathcal{G}^*(A)$$

holds true. So let us assume that $A \preceq^{\mathcal{G}^*} B$ and that B is not maximal relative to $\preceq^{\mathcal{G}^*}$. Recall that B is not maximal iff $B \in \Omega_{\mathcal{G}^*}$ and $\operatorname{rank}(B) < \min\{m, n\}$. We consider the following four exhaustive cases.

Case 1: $A \in \Omega_{\mathcal{G}}$, $B \in \Omega_{\mathcal{G}}$. This case is trivial because $\mathcal{G}$ is a property-p map.

Case 2: $A \in \Omega_{\mathcal{G}}$, $B \notin \Omega_{\mathcal{G}}$. Then $A \in \mathcal{UP}_{\mathcal{G}}(B)$, so that $\mathcal{G}^*(B) = \mathcal{G}_0(B)$. Since B is not maximal, $\mathcal{G}_0(B) \neq \emptyset$. Now let $B^- \in \mathcal{G}_0(B)$. By the definition of $\mathcal{G}_0(B)$,

$B^-AB^- \in \mathcal{G}(A)$. Since $A \in \Omega_{\mathcal{G}}$, $\mathcal{G}^*(A) = \mathcal{G}(A)$. Combining these observations results in $\mathcal{G}^*(A \mid B) \subseteq \mathcal{G}^*(A)$.

Case 3: $A \notin \Omega_{\mathcal{G}}$, $\mathcal{UP}_{\mathcal{G}}(A) = \emptyset$. Then $\mathcal{G}^*(A) = A\{1\}$. Since $A \preceq^{\mathcal{G}^*} B$, $B\{1\} \subseteq A\{1\}$ by Theorem 2.3(v). Then, in view of Theorem 2.1, $B^-AB^- \in A\{1,2\} \subseteq A\{1\}$ for each B^-. Observing that $\mathcal{G}^*(B) \subseteq B\{1\}$ thus yields $\mathcal{G}^*(A \mid B) \subseteq \mathcal{G}^*(A)$.

Case 4: $A \notin \Omega_{\mathcal{G}}$, $\mathcal{UP}_{\mathcal{G}}(A) \neq \emptyset$. Then $\mathcal{G}^*(A) = \mathcal{G}_0(A) \neq \emptyset$ because $A \in \Omega_{\mathcal{G}^*}$. So by Lemma 5.6, $\mathcal{UP}_{\mathcal{G}}(A) \subseteq \mathcal{UP}_{\mathcal{G}}(B)$. Therefore,

$$\mathcal{G}^*(B) = \begin{cases} \mathcal{G}(B) & \text{if } B \in \Omega_{\mathcal{G}}, \\ \mathcal{G}_0(B) & \text{otherwise.} \end{cases}$$

Since B is not maximal, necessarily $\mathcal{G}^*(B) \neq \emptyset$. By the definition of $\mathcal{G}_0$ and since $\mathcal{G}$ is a property-p map,

$$\mathcal{G}^*(C \mid B) \subseteq \mathcal{G}(C) \quad \text{for each} \quad C \in \mathcal{UP}_{\mathcal{G}}(A). \tag{5.9}$$

Recall that

$$\mathcal{G}_0(A) := \{A^- \mid A^-CA^- \in \mathcal{G}(C) \quad \text{for each} \quad C \in \mathcal{UP}_{\mathcal{G}}(A)\}.$$

Now let $C \in \mathcal{UP}_{\mathcal{G}}(A)$. Then, by (5.9), $B^-CB^- \in \mathcal{G}(C)$. Observe that $C \preceq^{\mathcal{G}} A \preceq^{\mathcal{G}^*} B$ implies, by Theorem 2.3, $B\{1\} \subseteq A\{1\}$, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, and $\mathcal{N}(A) \subseteq \mathcal{N}(C)$. Therefore, $B^-AB^- \in A\{1\}$ and $B^-AB^-CB^-AB^- = B^-CB^-$. So $(B^-AB^-)C(B^-AB^-) \in \mathcal{G}(C) = \mathcal{G}^*(C)$, which implies $B^-AB^- \in \mathcal{G}^*(A)$, by the definition of $\mathcal{G}^*(A)$. Hence again $\mathcal{G}^*(A \mid B) \subseteq \mathcal{G}^*(A)$, and the proof is complete. □

Theorem 5.3 also admits a version that includes the possibly nonsquare case. Since the proof is nearly identical, it is omitted.

THEOREM 5.8. *Let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be property-p maps on the set of $m \times n$ matrices. Further, let $\mathcal{G}^*$ be defined in respect to $\mathcal{G}$ according to* (5.2). *If $\preceq^{\tilde{\mathcal{G}}}$ is a $\mathcal{G}$-based extension of $\preceq^{\mathcal{G}}$, then $\preceq^{\tilde{\mathcal{G}}}$ is finer than $\preceq^{\mathcal{G}^*}$.*

In other words, if $\mathcal{G}$ is a property-p map then $\preceq^{\mathcal{G}^*}$ as defined via (5.2) represents the *maximal* possible $\mathcal{G}$-based extension of $\preceq^{\mathcal{G}}$ in the set of all property-p relations. This shows, in particular, that if $\preceq^{\mathcal{G}^*}$ does not support each matrix, then it is impossible to find a property-p relation that is an extension of $\preceq^{\mathcal{G}}$ and supports the whole set of matrices (recall the convention regarding the full-rank matrices). Example 3.4 has shown that in the nonsquare case there are semicomplete $\mathcal{G}$-based partial orders which fail to be property-p relations. The question of how to obtain a maximal $\mathcal{G}$-based partial order extension in such a case remains unanswered in this paper. It is expected, however, that the geometry of g-inversion (see §2) might be helpful in finding an answer.

Our next theorem will provide us with a sufficient condition on $\preceq^{\mathcal{G}}$ under which its maximal possible $\mathcal{G}$-based extension $\preceq^{\mathcal{G}^*}$ supports each matrix. Let us call D a *maximal element* of $\mathcal{UP}_{\mathcal{G}}(A)$ (in respect to $\preceq^{\mathcal{G}}$) if $D \in \mathcal{UP}_{\mathcal{G}}(A)$ and there is no matrix $C \in \mathcal{UP}_{\mathcal{G}}(A)$ such that $C \neq D$ and $D \preceq^{\mathcal{G}} C$. If $\mathcal{UP}_{\mathcal{G}}(A)$ possesses a unique maximal element C, C is called the *greatest element* of $\mathcal{UP}_{\mathcal{G}}(A)$.

THEOREM 5.9. *Let $\mathcal{G}$ be a property-p map such that, for each matrix A, $\mathcal{UP}_{\mathcal{G}}(A)$ possesses a greatest element whenever $\mathcal{UP}_{\mathcal{G}}(A) \neq \emptyset$. Again, let $\mathcal{G}^*$ be defined in respect to $\mathcal{G}$ according to* (5.2). *The property-p relation $\preceq^{\mathcal{G}^*}$ then supports each matrix and is the maximal possible (partial order) extension of $\preceq^{\mathcal{G}}$ in the class of property-p relations.*

Proof. Recalling Theorem 5.7, we have only to show that, in the framework of our theorem, $\mathcal{G}^*$ supports each matrix. By the definition of $\mathcal{G}^*$, clearly $\mathcal{G}^*(A) = \emptyset$ only if $A \notin \Omega_{\mathcal{G}}$ is such that $\mathcal{UP}_{\mathcal{G}}(A) \neq \emptyset$. Let $A \notin \Omega_{\mathcal{G}}$, and let $\mathcal{UP}_{\mathcal{G}}(A) \neq \emptyset$. Notice that $\mathcal{UP}_{\mathcal{G}}(A)$ has a greatest element, say D. So

$$C \preceq^{\mathcal{G}} D \preceq^{\mathcal{G}} A \tag{5.10}$$

for each $C \in \mathcal{UP}_{\mathcal{G}}(A)$. In particular, $D \preceq^{\mathcal{G}} A$. Hence $D \preceq^{-} A$ which, in view of Theorem 2.3, also implies $(A-D) \preceq^{-} A$. But then $D_r^{-}(A-D) = 0$ and $(A-D)D_r^{-} = 0$ for some $D_r^{-} \in \mathcal{G}(D)$ as well as $(A-D)_r^{-}D = 0$ and $D(A-D)_r^{-} = 0$ for some $\{1,2\}$-inverse $(A-D)_r^{-}$ of $(A-D)$. With these observations in mind it is easy to check that $G := D_r^{-} + (A-D)_r^{-}$ is a $\{1,2\}$-inverse of A and that $GDG = D_r^{-}$. Put

$$\tilde{\mathcal{G}}(A) := \{A^{-} \mid A^{-}DA^{-} \in \mathcal{G}(D)\}.$$

Since $G \in \tilde{\mathcal{G}}(A)$, $\tilde{\mathcal{G}}(A) \neq \emptyset$. In order to complete the proof it thus suffices to show that $\tilde{\mathcal{G}}(A) = \mathcal{G}^{*}(A)$. Trivially, $\tilde{\mathcal{G}}(A) \supseteq \mathcal{G}^{*}(A)$. So let $A^{-} \in \tilde{\mathcal{G}}(A)$ and let $C \preceq^{\mathcal{G}} A$. Notice that (5.10) implies $A\{1\} \subseteq D\{1\}$, $\mathcal{R}(C) \subseteq \mathcal{R}(D)$, and $\mathcal{N}(D) \subseteq \mathcal{N}(C)$ (recall Theorem 2.3). But then $A^{-}CA^{-} = A^{-}DA^{-}CA^{-}DA^{-}$. By observing $A^{-}DA^{-} \in \mathcal{G}(D)$, and since $\mathcal{G}(C \mid D) \subseteq \mathcal{G}(C)$, we now obtain $A^{-}CA^{-} \in \mathcal{G}(C)$, yielding that $A^{-} \in \mathcal{G}^{*}(A)$. □

At this point it is interesting to mention that Theorem 5.5 is in accordance with Theorem 5.9. For observe that, according to Lemma 2.1 in [7], $B \preceq^{\#} A$ implies $B \preceq^{\#} C_A$, so that the core part C_A of A is the *greatest* predecessor of A in respect to the sharp order. Theorem 5.5 tells us that $\mathcal{G}^{*}(A)$ can be defined, equivalently, by $\mathcal{G}_{*}(A)$ from (4.4), that is, in terms of the greatest (and so uniquely determined maximal) predecessor of A. Since this might be advantageous computationally it is pertinent to mention that for each property-p map $\mathcal{G}$ the crucial part $\mathcal{G}_0(B)$ in the definition of $\mathcal{G}^{*}(B)$ (see (5.1)) can always be redefined in a similar manner as

$$\mathcal{G}_0(B) := \{B^{-} \mid B^{-}CB^{-} \in \mathcal{G}(C) \quad \text{for each maximal element } C \text{ from } \mathcal{UP}_{\mathcal{G}}(B)\}. \tag{5.11}$$

The equivalence of these definitions can be seen basically as the last part in the proof of Theorem 5.9.

In context with the previous theorem it is further natural to ask the following: Is the phenomenon observed in Example 5.1 universally true whenever there is multiplicity of *maximal* elements for at least one set $\mathcal{UP}_{\mathcal{G}}(A)$? In other words, does failure of uniqueness always correspond to a situation where along with $\preceq^{\mathcal{G}}$ all its $\mathcal{G}$-based property-p extensions also have poor support? That this is not the case is illustrated in our final example.

Example 5.10. Consider the matrices

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Put $B := A_1 + A_2$. Observe that these matrices are all of index 1. Check $A_1^{\#} = A_1$ and $A_2^{\#} = A_2$. Define the map $\mathcal{G}(\cdot)$ on the set of 3×3 matrices according to

$$\mathcal{G}(A) = \begin{cases} \{A_1\} & \text{if } A = A_1, \\ \{A_2\} & \text{if } A = A_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Notice that $\preceq^{\mathcal{G}}$ defines a partial ordering. Checking $A_i \preceq^{\mathcal{G}} B$ $(i = 1, 2)$ yields $\mathcal{UP}_{\mathcal{G}}(B) = \{A_1, A_2\}$. The sharp order $\preceq^{\#}$ is obviously a possible partial order extension of $\preceq^{\mathcal{G}}$. It thus follows from Theorem 4.2 that there is a partial order extension of $\preceq^{\mathcal{G}}$ that supports each matrix, although $\mathcal{UP}_{\mathcal{G}}(B)$ does not possess a greatest element. □

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