ROOTS OF SYMMETRIC IDEMPOTENT BOOLEAN MATRICES *

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Abstract

Boolean matrices A such that A^m is symmetric and idempotent are characterized. As two of the applications, characterizations of Boolean matrices A such that for some positive integer m, (i) $A^\dagger = A^m$, or (ii) $A^m = A^{m+1}$, where A^m is symmetric, are derived.

1. INTRODUCTION

We consider the Boolean algebra $\{0,1\}$ equipped with the operations of Boolean addition and multiplication, which are the same as the usual operations, except that $1+1=1$. In this paper, we deal exclusively with Boolean matrices, i.e., matrices over the Boolean algebra $\{0,1\}$. Matrix addition and multiplication are defined in the usual way. For a comprehensive survey of Boolean matrices, we refer to [3].

A characterization of (entrywise) nonnegative matrices A such that A^m is symmetric and idempotent was obtained in [2]. In the present paper, we consider a similar problem for Boolean matrices. We first introduce some notation. As usual the group of permutations of $\{1,2,\cdots,n\}$ will be denoted by S_n. J_ℓ will denote the $\ell \times \ell$ matrix of all ones and if the order of the matrix is not relevant, then we will simply denote it by J. For a matrix A, A^T and $A^\dagger$ will denote the transpose of A and the Moore-Penrose inverse of A, respectively. If $A_1,\cdots,A_k$ are matrices, then $A_1 \oplus \cdots \oplus A_k$ will denote their direct sum

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

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The main result gives a complete characterization of matrices A such that A^m is symmetric and idempotent.

2. THE MAIN RESULT

THEOREM 1: Let A be an $n \times n$ matrix. Then A^m is symmetric and idempotent if and only if there exists a permutation matrix Q such that QAQ^T is a direct sum of square matrices of the following (not necessarily all) three types.

(I) C_{11}, where $C_{11}^m = J$

(II) $$\begin{bmatrix} 0 & C_{12} & 0 & \cdots & 0 \\ 0 & 0 & C_{23} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & C_{d-1\,d} \\ C_{d1} & 0 & \cdot & \cdots & 0 \end{bmatrix}$$

where d | m, $d \neq 1$, the zeros on the diagonal are square matrices of appropriate order and

$$(C_{jj+1}C_{j+1j+2}\cdots C_{j+d-1j})^{m/d} = J, \quad j = 0, 1, \cdots, d$$

where the subscripts are to be interpreted modulo d.

(III) A block matrix $[C_{i,j}]$, $i, j = 1, 2, \cdots, \ell$ where $\ell \leq m$, $C_{ij} = 0$ if $i \geq j$ and C_{ii} is square.

Before proving Theorem 1, we obtain some preliminary results. The proofs of the following lemmas are similar to those of Lemmas 4-6 in [2] and hence are omitted.

LEMMA 1: Suppose $C^m = D$ where C, D are $n \times n$ matrices conformally partitioned as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

where the diagonal blocks are square and each diagonal entry of D_{11} is 1. Then C_{12}, C_{21} are zero matrices, $C_{11}^m = D_{11}$ and $C_{21}^m = 0$.

LEMMA 2: Let C be an $n \times n$ matrix such that $C^m = 0$. Then there exists a permutation matrix Q such that QCQ^T is the block matrix $[C_{ij}]$, $i, j = 1, 2, \cdots, \ell$ where $\ell \leq m$, $C_{ij} = 0$ if $i \leq j$ and C_{ii} is square.

LEMMA 3: Let $C = [C_{ij}]$, $i, j = 1, 2, \cdots, n$ be a block matrix where C_{ij} is $\ell_i \times \ell_j$ and suppose $C^m = J_{\ell_1} \oplus \cdots \oplus J_{\ell_n}$. Then there exists $\sigma \in S_n$ such that

(a) $C_{j\ell}$ is the zero matrix except when $\ell = \sigma(j)$

(b) $(C_{j\sigma(j)} C_{\sigma(j)\sigma^2(j)} \cdots C_{\sigma^{d_j-1}(j)j})^{m/d_j} = J_{\ell_j}$

where d_j is the least positive interger such that $\sigma^{d_j}(j) = j$.

Consequently, there exists a permutation matrix Q such that QCQ^T is a direct sum of square matrices of types (I) and (II) (not necessarily both types) given in Theorem 1.

Proof of Theorem 1. The "if" part is easy. To prove the "only if" part, note that since A^m is symmetric, idempotent, there exists a permutation matrix Q such that $QAQ^T = J_{\ell_1} \oplus \cdots \oplus J_{\ell_m} \oplus 0$. This is seen by first reducing A^m to the Frobenins Normal Form and then using the well-known fact (see [1], for example) that an irreducible symmetric idempotent matrix must be equal to J. The proof now follows from Lemmas 1, 2 and 3.

3. APPLICATIONS

THEOREM 2: Let A be an $n \times n$ matrix. Then A^m is symmetric and $A^{m+1} = A$ if and only if there exists a permutation matrix Q such that QAQ^T is a direct sum of the following (not necessarily both) two types.

(i) A square matrix of all ones or all zeros

$$\text{(ii)} \quad \begin{bmatrix} 0 & J_{\ell_1} & 0 & \cdots & 0 \\ 0 & 0 & J_{\ell_2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdot & \cdots & J_{\ell_{d-1}} \\ J_{\ell d} & \cdot\cdot & \cdots & \cdots & 0 \end{bmatrix}$$

where $d \mid m$.

Proof: Note that A^m is idempotent. Therefore, by Theorem 1 there exists a permutation matrix Q such that QAQ^T is a direct sum of square matrices of type (I), (II) or (III). Since $A^{m+1} = A$, each summand S of QAQ^T satisfies $S^{m+1} = S$. If S is of type (I), then $S^m = J$. Thus S cannot have a zero row and, therefore, $S = S^{m+1} = SJ = J$. If S is of type (II), then using a similar argument we can show that S must be of type (ii). Finally, if S is of type (III), then $S^m = 0$. Thus $S = 0$.

REMARK: As an immediate consequence of Theorem 2, we see that if A is an $n \times n$ matrix such that $A^\dagger$ exists (in which case it, in fact, is A^T, see for example [4]) and if $A^\dagger = A^m$, then there exists a permutation matrix Q such that QAQ^T is a direct sum of square matrices of types (i) or (ii).

As another application, we can obtain the characterization of Boolean matrices A such that A^m is symmetric and $A^{m+1} = A^m$ in the following theorem.

THEOREM 3: Suppose A is a Boolean matrix such that $A^m = A^{m+1}$ and A^m is symmetric. Then there exists a permutation matrix Q such that QAQ^T is a direct sum of matrices of the following types:

(i) A square matrix of all ones or all zeros.

(ii) A nilpotent matrix partitioned as $\ell \times \ell$ block matrix $[C_{ij}]$ where $C_{ij} = 0$ if $i \geq j$ and C_{ii} is square, $\ell \leq m$.

Proof. Follows from Theorem 1.

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