On the Periodicity of the Graph of Nonnegative Matrices

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ABSTRACT

We study nonnegative matrices A such that $\Gamma(A) = \Gamma^n(A)$. As a consequence we obtain Flor's characterization of nonnegative idempotent matrices and other well-known results.

1. DEFINITIONS AND NOTATION

A matrix $A = (a_{ij})$ is called nonnegative if $a_{ij} \geqslant 0$ for all i, j, and a matrix A is called positive if $a_{ij} > 0$ for all i, j. To denote nonnegative and positive matrices, we write $A \geqslant 0$ and $A > 0$, respectively.

As usual, we define the (directed) graph of an $m \times m$ matrix $A = (a_{ij})$ to be the graph $\Gamma(A)$ with vertices $1, 2, \dots, m$, where (i, j) is an edge if and only if $a_{ij} \neq 0$.

A *path* of length n is a sequence of n edges $(i_0, i_1), (i_1, i_2), \dots, (i_{n-1}, i_n)$ in which the terminal vertex of one edge is the initial vertex of the next. A directed graph $\Gamma(A)$ of a matrix A is *strongly connected* if for any ordered pair (i, j) of vertices of $\Gamma(A)$, there exists a path which leads from i to j. An edge (i, i) is called a loop. If the graph of a matrix is strongly connected, then the matrix is called *irreducible*. A matrix A is called *primitive* if some power of A is positive. When a matrix A is in the block form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1p} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & A_{p3} & \cdots & A_{pp} \end{pmatrix},$$

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we define the block graph of A to be the graph $\Gamma_B(A)$ with vertices $1, 2, \ldots, p$, where (i, j) is an edge if and only if $A_{ij} \neq 0$.

Given a matrix A, a maximal strongly connected subgraph of the graph of A is called a strong component of the graph of A. Vertices which do not belong to any strong component are called singular vertices. Given a nonnegative matrix A, we may relabel the strong components and singular vertices to obtain a matrix which is in the familiar (lower triangular) Frobenius normal form. Relabeling of strong and singular components is equivalent to the existence of a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & A_{p3} & \cdots & A_{pp} \end{pmatrix},$$

where each diagonal block A_{ii} is irreducible or a 1×1 zero matrix. The 1×1 zero matrices correspond to singular vertices in the graph of A. Given a nonnegative matrix A and a permutation matrix P, the graph of PAP^T is a relabeling of the vertices of the original graph of A. Hence for any two matrices A and B, $\Gamma(A) = \Gamma(B)$ implies $\Gamma(PAP^T) = \Gamma(PBP^T)$ for any permutation matrix P.

A block matrix (of size $h \times h$) of the form

$$A = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{h-1h} \\ A_{h1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $A_{h1} \neq 0$ and each $A_{ii+1} \neq 0$ for $i = 1, \ldots, h-1$, is called a *cyclic matrix of index h*. If, in addition, A_{h1} and each A_{ii+1} is positive, then the matrix A will be called a *complete cyclic matrix of index h*, and its associated graph $\Gamma(A)$ will be called a *complete cyclically h-partite graph*. Note that the partitioning of the matrix induces a partition of the set of vertices into subsets $P^{(1)}, P^{(2)}, \ldots, P^{(h)}$.

If Γ_1 and Γ_2 are graphs, then the product graph $\Gamma_1\Gamma_2$ is defined as follows: $(i, j) \in \Gamma_1\Gamma_2$ if there is a $k \in V = \{1, 2, \ldots, n\}$ such that $(i, k) \in \Gamma_1$ and $(k, j) \in \Gamma_2$. We write $\Gamma^2 = \Gamma\Gamma$, $\Gamma^3 = \Gamma^2\Gamma$, and so on. Let $\Delta =$

$\{(i,i): i \in V\}$. The reflexive transitive closure $\bar{\Gamma}$ of a graph Γ is defined by

$$\bar{\Gamma} = \Delta \cup \Gamma \cup \Gamma^2 \cup \cdots.$$

Thus, $(i,j) \in \bar{\Gamma}$ if and only if there is a path from i to j in Γ.

The following is a simple fact about nonnegative matrices A and B [9].

Proposition 1. *For $c \neq 0$, $\Gamma(cA) = \Gamma(A)$; $\Gamma(A+B) = \Gamma(A) \cup \Gamma(B)$; $\Gamma(AB) = \Gamma(A)\Gamma(B)$. Therefore, if $A \geqslant 0$, $\Gamma(A^n) = \Gamma(A)\Gamma(A)\cdots\Gamma(A)$ (n factors). Thus $\Gamma(A^n) = \Gamma^n(A)$.*

2. PRELIMINARY RESULTS

Sublemma 2.1. *Let A be a nonnegative primitive matrix. Then for $n > 1$, $\Gamma(A) = \Gamma^n(A)$ if and only if $A > 0$.*

The proof of this sublemma is obvious.

Lemma 2.2. *Let A be a nonnegative irreducible matrix which is not primitive, such that $\Gamma(A) = \Gamma^n(A)$ for some $n > 2$. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{h-1h} \\ A_{h1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $A_{h1} > 0$ and $A_{i-1i} > 0$, $i = 2,\ldots,h$, and $n \equiv 1 \pmod h$, that is, $\Gamma(A)$ is a complete cyclically h-partite graph.

Proof. Since A is not primitive, there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{h-1h} \\ A_{h1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $A_{h1} \neq 0$ and $A_{i-1i} \neq 0$, $i = 2\dots, h$. We first observe that $n \equiv 1 \pmod h$, since $n \not\equiv 1 \pmod h$ along with $\Gamma(A) = \Gamma^n(A)$ implies $A = 0$. Write $n = qh + 1$ for $q \geqslant 1$. Then

$$PA^nP^T = PA^{qh+1}P^T = \begin{pmatrix} 0 & C_1^qA_{12} & 0 & \cdots & 0 \\ 0 & 0 & C_2^qA_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_{h-1}^qA_{h-1h} \\ C_h^qA_{h1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $C_i = A_{ii+1}A_{i+1i+2}\cdots A_{i-1i}$ (indices taken mod h) and each C_i is primitive [2]. Hence $\Gamma(A) = \Gamma^n(A)$ if and only if $C_i^qA_{ii+1}$; and A_{ii+1} have the same zero pattern for all $i = 1,2,\dots,h$ (indices taken mod h). This implies $C_i^{qj}A_{ii+1}$ and A_{ii+1} also have the same zero pattern for all $j > 0$. Since C_i is primitive, $C_i^r > 0$ for all $r \geqslant s$ for some s. Now choose some j such that C_i^{qj} is positive. Since A is irreducible, A_{ii+1} has no zero rows and no zero columns. Hence $C_i^{qj}A_{ii+1}$ is positive, and so it follows that A_{ii+1} is positive, $i = 1,2,\dots,h-1,h$ (indices taken mod h). This completes the proof. ■

REMARK 2.3. If A is a nonnegative irreducible matrix such that $\Gamma(A) = \Gamma^2(A)$, then A is positive.

LEMMA 2.4. *Let α, β, and n be positive integers with $n > 1$, $\alpha \mid n-1$ and $\beta \mid n-1$, and $d = \gcd(\alpha,\beta)$. Then there exist $\lambda,\mu,\lambda',\mu' > 0$ such that $n = \lambda\alpha + \mu\beta + d + 1$ and $n = \lambda'\alpha + \mu'\beta - d + 1$.*

Proof. Write $\alpha/d = \alpha'$ and $\beta/d = \beta'$, then clearly $(\alpha',\beta') = 1$. This, together with $\alpha' \mid (n-1)/d$ and $\beta' \mid (n-1)/d$ implies $\alpha'\beta' \mid (n-1)/d$, so that $(n-1)/d \geqslant \alpha'\beta'$ and hence $(n-1)/d - 1 \geqslant \alpha'\beta' - 1 \geqslant (\alpha'-1)(\beta'-1)$.

Since any positive integer $\geqslant (\alpha'-1)(\beta'-1)$ is an integral nonnegative linear combination of α' and β', there exist $\lambda,\mu \geqslant 0$ such that $\lambda\alpha' + \mu\beta' = (n-1)/d - 1$. [See A. Brauer, *Amer. J. Math.* 64:302 (1942), Corollary.] Thus, $\lambda\alpha + \mu\beta + d = n - 1$. Similarly, $n = \lambda'\alpha + \mu'\beta - d + 1$. ■

REMARK 2.5. Assume the hypothesis in Lemma 2.4. If $n - t = \lambda\alpha + \mu\beta$ where $\lambda,\mu \geqslant 0$ and $t \geqslant 1$, then $t = ld + 1$ for some $l \geqslant 0$.

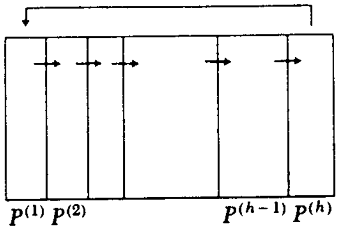

FIG. 1

Let $A=(A_{ij})$ be a matrix in Frobenius form, and let A_{ii} and A_{jj} be cyclic matrices of index h_i and h_j, respectively, for some i and j. If P_i and P_j are the subsets of vertices of $\Gamma(A)$ corresponding to the cyclic matrices A_{ii} and A_{jj}, then (as defined earlier) $P_i^{(l)}$, $l=1,\dots,h_i$, and $P_j^{(m)}$, $m=1,\dots,h_j$, will denote the partitioning of the sets P_i and P_j induced by the cyclicity of A_{ii} and A_{jj}. The sets P_i and P_j will be referred to as classes of types h_i and h_j, and $P_i^{(l)}$ and $P_j^{(m)}$) as their subclasses in $\Gamma(A)$, respectively. If A_{ii} is positive, then we say P_i is a complete class.

A complete cyclically h-partite graph can be represented by a diagram as in Figure 1, where $\rightarrow$ indicates that there is an edge from any vertex in $P^{(k)}$ to any vertex in $P^{(k+1)}$, $k=1,\dots,h$ (superscripts mod h).

LEMMA 2.6. *Let $A\geqslant 0$, with $\Gamma(A)=\Gamma^n(A)$. Let $PAP^T=(A_{ij})$ be a Frobenius form of A such that A_{ii} and A_{jj} are cyclic of index h_i and h_j, respectively, with P_i and P_j being the classes of types h_i and h_j corresponding to A_{ii} and A_{jj}. If there is an edge from $P_i^{(l)}$ to $P_j^{(m)}$, $1\leqslant l\leqslant h_i$, $1\leqslant m\leqslant h_j$, then for each $x\in P_i^{(l)}$ and $y\in P_j^{(m)}$, (x,y) is an edge in $\Gamma(A)$.*

Proof. It is clear from Remark 2.3 that $n>2$. Now from Lemma 2.2, A_{ii} is a complete cyclic matrix of index h_i and $h_i\mid n-1$. Let $r\in P_i^{(l)}$, $s\in P_j^{(m)}$ such that $(r,s)\in\Gamma(A)$. For $x\in P_i^{(l)}$, it is easy to determine from the diagram in Figure 2 that there exists a path of length kh_i+1 from x to s for all $k\geqslant 1$. Since $h_i\mid n-1$, there exists a path of length n from x to s. Hence, there is an edge from x to s. Similarly, since there is an edge from x to s, for each $y\in P_j^{(m)}$, there is a path of length n (hence an edge) from x to y. ■

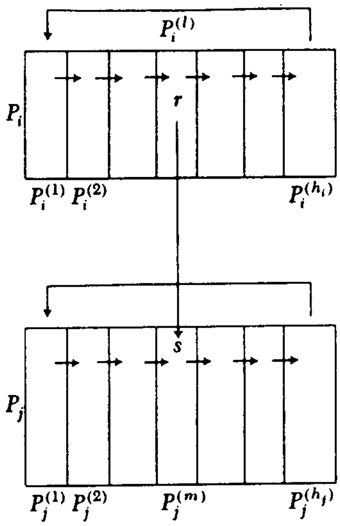

FIG. 2

3. MAIN RESULT

THEOREM 3.1. *Suppose A is a nonnegative matrix such that $\Gamma(A) = \Gamma^n(A)$. Let $PAP^T = [A_{ij}]_{p \times p}$ be a Frobenius form of A where P is a permutation matrix. If $A_{ii} \neq 0$ for all $i = 1, 2, \ldots, p$, then*

(i) $A_{ii} > 0$ or A_{ii} *is a complete cyclic matrix of index h where $n \equiv 1 \pmod{h}$;*

(ii) *for $j < i$, $i = 1, \ldots, p$, and for the blocks A_{ii} and A_{jj} of sizes $h_i \times h_i$ and $h_j \times h_j$, respectively, A_{ij} is a direct sum of matrices of the type*

$$\begin{pmatrix}
0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 \\
* & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots \\
0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 \\
\cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 \\
\cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * \\
0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 \\
* & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0 & \cdots \\
0 & * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & * & 0
\end{pmatrix},$$

where the distance between any two consecutive nonzero diagonals is $d_{ij} = \gcd(h_i, h_j)$ *and asterisks represent positive blocks; and*

(iii) *the block graph of* A *is transitively closed.*

Proof. (i): Since $\Gamma(A) = \Gamma^n(A)$, $\Gamma(A_{ii}) = \Gamma^n(A_{ii})$; thus the result follows from Lemmas 2.2 and 2.1.

(ii): Suppose A_{ii} is cyclic of index h_i and A_{jj} is cyclic of index h_j. Then A_{ii} and A_{jj} corresponds to classes P_i, P_j with subclasses $P_i^{(1)}, P_i^{(2)}, \dots, P_i^{(h_i)}$ and $P_j^{(1)}, P_j^{(2)}, \dots, P_j^{(h_j)}$, respectively. Now suppose $A_{ij} \neq 0$; then there are $r \in P_i^{(l)}$, $s \in P_j^{(m)}$ such that $(r, s) \in \Gamma(A)$. Now we show this implies the existence of edges from $P_i^{(l)}$ to $P_j^{(m+d_{ij})}$ and $P_i^{(l)}$ to $P_j^{(m-d_{ij})}$, $1 \leqslant m \pm d_{ij} \leqslant h_j$. Since (r, s) is an edge in $\Gamma(A)$, for all $\lambda, \mu > 0$ there is a path from $P_i^{(l)}$ to $P_j^{(m+d_{ij})}$ of length $\lambda h_i + \mu h_j + d_{ij} + 1$. Consequently, by Lemma 2.4, there is a path of length n from $P_i^{(l)}$ to $P_j^{(m+d_{ij})}$, and so there is an edge from $P_i^{(l)}$ to $P_j^{(m-d_{ij})}$, since $\Gamma(A) = \Gamma^n(A)$. Since there is an edge from $P_i^{(l)}$ to $P_j^{(m+d_{ij})}$, for all $\lambda, \mu > 0$ there is a path of length $\lambda h_i + \mu h_j - d_{ij} + 1$ from $P_i^{(l+1)}$ to $P_j^{(m+1)}$. By Lemma 2.4, there is a path of length n (hence an edge) from $P_i^{(l+1)}$ to $P_j^{(m+1)}$. Similarly we can obtain an edge from $P_i^{(l-1)}$ to$P_j^{(m-1)}$. Our argument shows that if there is an edge from $P_i^{(l)}$ to $P_j^{(m)}$, then there exist edges from

(a) $P_i^{(l)}$ to $P_j^{(m \pm d_{ij})}$,
(b) $P_i^{(l \pm 1)}$ to $P_j^{(m \pm 1)}$.

This translates to: If $(A_{ij})_{l,m} \neq 0$ then

(a′) $(A_{ij})_{l, m \pm d_{ij}} \neq 0$,
(b′) $(A_{ij})_{(l \pm 1),(m \pm 1)} \neq 0$.

By repeating the same argument with the new nonzero entries of A_{ij}, we obtain A_{ij} as a direct sum of matrices of the desired form. By Lemma 2.6, each nonzero block is a positive block. Lemma 2.5 guarantees that the edge (r, s) in $\Gamma(A)$ will not give rise to any edge other than the ones described above. Notice when $h_i = 1$ or $h_j = 1$, then A_{ii} or A_{jj} is a positive matrix. In this case $d_{ij} = \gcd(h_i, h_j) = 1$, and so $A_{ij} \neq 0$ implies $A_{ij} > 0$.

(iii): To show $\Gamma_B(A)$ is transitively closed, suppose $A_{ij} \neq 0$ and $A_{jk} \neq 0$. Then both A_{ij} and A_{jk} must be sums of matrices of the type described in (ii). Therefore, $A_{ii}^{n-2}A_{ij}A_{jk}$ is a nonzero sum of matrices of the type described in (ii). Since a nonzero (u, v) entry in $A_{ii}^{n-2}A_{ij}A_{jk}$ implies a nonzero (u, v) entry in A_{ik}, it follows that $A_{ik} \neq 0$. Hence $\Gamma_B(A)$ is transitively closed. ∎

REMARK. It is possible to obtain similar results about the structure of the Frobenius form without the assumption that $A_{ii} \neq 0$ for all i. However, there are very many cases that need to be described.

COROLLARY 3.2. *Suppose A is a nonnegative matrix and $PAP^T = [A_{ij}]_{p\times p}$ is a Frobenius form of A, where P is a permutation matrix. If $A_{ii} \neq 0$ for all $i = 1,2,\dots,p$, then $\Gamma(A) = \Gamma^2(A)$ if and only if $A_{ii} > 0$ $(i = 1,2,\dots,p)$, A_{ij} is zero or positive for $j < i$, and the block graph of A is transitively closed.*

COROLLARY 3.3. *Assume the hypothesis in Corollary 3.2. Then $\Gamma(A) = \Gamma^3(A)$ implies:*

(i) *$A_{ii} > 0$ or A_{ii} is a complete cyclic matrix of index 2;*

(ii) *for $j < i$, A_{ij} is either*

(a) *zero,*

(b) *positive,*

(c) *of the form* $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$, *or*

(d) *of the form* $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$;

(iii) *the block graph of A is transitively closed.*

REMARK 3.4. It is natural to ask whether the converse of the theorem is true. The answer is in the affirmative if $n = 2$, as we have seen in Corollary 3.2. However, for $n \geqslant 3$, further restrictions on the block structure of A must be made. For example, in order to obtain sufficiency for the case $n = 3$, the following must be included along with conditions (i)–(iii) above:

(iv) If $A_{ij} > 0$, then either class P_i has access to class P_j via a complete class in the block graph, or there exists l, k, l', k' such that $A_{ik}A_{kl}A_{lj}$ is of the form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $A_{ik'}A_{k'l'}A_{l'j}$ is of the form $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, where each $*$ represents a positive matrix (possibly rectangular).

(v) If A_{ij} is of the form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$, then for all k, l, either $A_{ik}A_{kl}A_{lj}$ is zero or it is of the same form.

(vi) If A_{ij} is of the form $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, then for all k, l, either $A_{ik}A_{kl}A_{lj}$ is zero or it is of the same form.

We give an example of a matrix A with $\Gamma(A) = \Gamma^5(A)$ to illustrate the structure described in Theorem 3.1.

EXAMPLE 3.5. Let

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$A_{21} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We note the following:

(i) A_{11}, A_{22} are cyclic matrices of index 4, 2, respectively, and $5 \equiv 1 \pmod{h}$, where $h = 4$ or 2.

(ii) The distance between any two nonzero consecutive diagonals of A_{21} is 2, which is the gcd of 4 and 2.

(iii) The block graph of A is as shown in Figure 3, and is transitively closed.

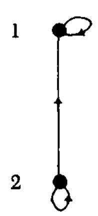

FIG. 3

4. APPLICATIONS

The following is perhaps well known. However, we are not aware of an elementary proof which does not depend upon some other important result.

LEMMA 4.1. *Let E be a nonnegative idempotent irreducible matrix. Then* rank $E = 1$, *and so* $E = xy^T$, $x > 0$, $y > 0$, $y^T x = 1$.

Proof. The spectral radius of E is 1, and by the Perron-Frobenius theorem, 1 is a simple eigenvalue of E. That is, 0 is a simple eigenvalue of $1 - E$. This implies that the dimension of the null space of $(I - E)^T$ is 1. The rows of E are eigenvectors of $(I - E)^T$ corresponding to eigenvalue 0, since $(I - E)^T E^T = 0$. This proves that the dimension of the row space of E is 1. Hence rank $E = 1$, so $E = xy^T$, $x, y \geqslant 0$. Since E is irreducible, it follows that $E > 0$. ■

THEOREM 4.2 [2]. *Let A be a nonnegative idempotent matrix with no zero rows and no zero columns. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} x_1 y_1^T & 0 & \cdots & 0 \\ 0 & x_2 y_2^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x_p y_p^T \end{pmatrix}$$

where $x_i, y_i > 0$ and $y_i^T x_i = 1$.

Proof. We know that there exists a permutation matrix P such that

$$PAP^t = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{pmatrix}.$$

Since there are no zero rows and no zero columns, $A_{11} \neq 0$ and $A_{pp} \neq 0$. Suppose $A_{ii} = 0$ for some i, $2 \leqslant i \leqslant p - 1$. Let k be the largest integer such that $A_{kk} = 0$, $2 \leqslant k \leqslant p - 1$. If $A_{lk} = 0$ for all $l = k + 1, k + 2, \ldots, p$, then the kth column of A is zero, which is not true under our hypothesis. Therefore, $A_{lk} \neq 0$ for some $l = k + 1, k + 2, \ldots, p$. Then, since $A = A^2$, it follows that

A_{lk} is positive. For $l \geqslant k+1$ we have $A_{ll} > 0$ by the choice of k. Since A_{ll} is idempotent and irreducible, by Lemma 4.1, $A_{ll} = x_l y_l^T$, where $y_l^T x_l = 1$ and $x_l > 0$, $y_l > 0$. Then $A = A^2$ yields

$$A_{lj} = A_{ll}A_{lj} + \sum_{j \leqslant i < l} A_{li}A_{ij}$$

$$= x_l y_l^T A_{lj} + \sum_{j \leqslant i < l} A_{li}A_{ij}.$$

Multiplying this equation on the left by y_l^T, we have $y_l^T(\Sigma_{j \leqslant i < l} A_{li}A_{ij}) = 0$. Since $y_l > 0$, this implies $A_{li}A_{ij} = 0$ for all $i = j, j+1, \dots, l-1$. In particular, when $i = k$, $A_{lk}A_{kj} = 0$ for $j = 1, 2, \dots, k-1$. Since $A_{lk} > 0$, this implies $A_{kj} = 0$ for all $j = 1, 2, \dots, k-1$. That is, the kth row of A is zero, a contradiction. Therefore, $A_{ii} \neq 0$ for all i. Similarly, the equation $A_{ij} = A_{ii}A_{ij} + A_{ij}A_{jj} + \Sigma_{j<k<i} A_{ik}A_{kj}$ yields $A_{ij} = 0$ for all $j < i$, since $A_{ii} = x_i y_i^T$ where $y_i^T x_i = 1$. ∎

REMARK 4.3. If A is any matrix, then there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} J & K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L & M & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

where J, K have no zero rows in common, and J, L have no zero columns in common. In case A is idempotent, it immediately follows that $K = JD$, $L = CJ$, $M = CJD$, where J is an idempotent matrix with no zero rows and no zero columns. Thus by the above theorem the well-known characterization of any nonnegative idempotent matrix may be obtained.

REMARK 4.4. By applying Theorem 3.1 we can also obtain the well-known characterizations of nonnegative matrices A such that $A = A^n$ [1, 5, 7, 8].

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