# New Classes of Monotone Matrices

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ABSTRACT

In this note we introduce new classes of matrices having nonnegative inverses.

## 1. INTRODUCTION

The investigation of classes of monotone matrices, for example M-matrices, has been of interest to many authors in view of the vast applications in which monotone matrices play an important role [1]. Among the many known results in the study of monotone matrices are conditions for the total nonnegativity of the inverse of a Toeplitz matrix with a positive diagonal in terms of the roots of a certain polynomial [3]. Theorem 4 of this paper concerns a class of monotone symmetric Toeplitz matrices which have a negative diagonal. Theorems 1 and 2 each yields a class of monotone triangular Toeplitz matrices.

The proofs of Theorems 1 and 2 depend upon several computational lemmas. The proof of Theorem 4 uses Gerschgorin's theorem and the interlacing theorem for eigenvalues of perturbed symmetric matrices.

## 2. NOTATION AND TERMINOLOGY

For $d \geq 0$ and $k = 1,2,\ldots,n$, define $a_k = (-1)^{k-1}(a+(k-1)d)$. Let A denote a Toeplitz matrix of the form


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$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_1 & a_2 & \cdots & a_{n-1} \\ a_3 & a_2 & a_1 & \cdots & a_{n-2} \\ & & \vdots & & \\ a_n & a_{n-1} & \cdots & & a_1 \end{bmatrix}$$

We will call such an A, a symmetric arithmetic progression matrix or symmetric APM. For

$$T = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ & & \vdots & & \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n$ are as above, we will call T a triangular arithmetic progression matrix or a triangular APM. Note that when T is a triangular APM, we can express $T = a_1 I + a_2 S + \cdots + a_n S^{n-1}$, where

$$S = \begin{bmatrix} 0 & 0 & & \cdots & & 0 \\ 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Since there is no loss of generality for the results to be obtained if we assume $a_1 = 1$, we shall do so throughout. $e_{i,j}$ will denote a square matrix with $(i,j)$ entry equal to 1 and zeros elsewhere. A nonsingular matrix A is called inverse monotone or simply monotone if $A^{-1} \geq 0$, that is, each entry of the inverse of A is nonnegative.

## 3. TRIANGULAR APM AND PERTURBATIONS

In this section we first determine necessary and sufficient conditions for a triangular APM to have a nonnegative inverse.

LEMMA 1. Let A be an nxn triangular APM.

(a) For $2 \leq n \leq 3$, $A^{-1} \geq 0$ if and only if $d \geq -1$.

(b) For $n \geq 4$, $A^{-1} \geq 0$ if and only if $d \geq 1$ or $d = 0$.

Proof: From the representation $A = I + \sum_{k=1}^{n-1} (-1)^k (1+kd) S^k$, it is easily verified that

$$(I+S)^2 A = I - (d-1)S, \text{ and so}$$
$$A^{-1} = (I - (d-1)S)^{-1}(I+S)^2.$$

It follows that

$$A^{-1} = I + (d+1)S + d^2S^2 + (d-1)d^2S^3 + \ldots + (d-1)^{n-3}d^2S^{n-1}.$$

Parts (a) and (b) are a direct consequence of this representation for $A^{-1}$. □

LEMMA 2. Let A be a triangular APM with $d \geq 1$ or $d = 0$.

(a) If $H \geq 0$ and is strictly lower triangular, then
$$(A - H)^{-1} \geq 0.$$

(b) If D is a diagonal matrix with $0 \leq D < I$, then
$$(A - D)^{-1} \geq 0.$$

Proof: (a) $(A - H)^{-1} = A^{-1}(I - HA^{-1})^{-1}$
$$= A^{-1}(I + HA^{-1} + (HA^{-1})^2 + \ldots + (HA^{-1})^{n-1}),$$
since $(HA^{-1})^n = 0$. It follows that $(A - H)^{-1} \geq 0$ since $A^{-1} \geq 0$ by Lemma 1. □

The proof of part (b) is similar.

REMARKS. If A is a triangular APM with $A^{-1} \geq 0$ and if H is a nonnegative triangular matrix for which the spectral radius of $HA^{-1}$ is less than 1, then it follows that $(I - HA^{-1})^{-1} \geq 0$; consequently $(A - H)^{-1} \geq 0$.

If A is a triangular APM with $A^{-1} \geq 0$ and if H and D are as in Lemma 2, then $(A - D - H)^{-1} \geq 0$.

THEOREM 1. Let $\{a_k\}$ be an increasing sequence of positive real numbers such that there is some $d \geq 1$ for which $a_k - a_{k-2} \leq 2d$ when k is odd and $a_k - a_{k-2} \geq 2d$ when k is even. Furthermore, assume $a_1 = 1$ and $a_2 \geq 1 + d$. If $T = (t_{ij})$ is a lower triangular nxn Toeplitz matrix with $t_{i1} = (-1)^{i-1}a_i$, $1 \leq i \leq n$, then T is monotone.

Proof: Suppose d satisfies the conditions in Theorem 1. From the conditions on the sequence $\{a_k\}$, it follows that $a_k \leq 1 + (k-1)d$ if k is odd and $a_k \geq 1 + (k-1)d$ if k is even. Thus, if

$$A = I + \sum_{k=1}^{n-1} (-1)^k (1+kd) S^k$$ is a triangular APM and $H = A - T$,

then $H \geq 0$ and H is strictly lower triangular. Therefore, it follows from Lemma 2 that $T = A - H$ is monotone. □

If A is a triangular APM such that $A^{-1} \geq 0$, then $(A - \alpha e_{i,j})^{-1} \geq 0$, for all $i,j = 1,2,\ldots,n$ and $i > j$, where $\alpha > 0$ (Lemma 2). However, it is easy to see that $(A + \alpha e_{i,j})^{-1}$ need not be nonnegative. For example, if A is a 4x4 triangular APM with d = 1, then

$$(A + \alpha e_{2,1})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2-\alpha & 1 & 0 & 1 \\ 1-2\alpha & 2 & 1 & 0 \\ -\alpha & 1 & 2 & 1 \end{bmatrix}.$$

Theorem 2 will show that it is possible to add $\alpha > 0$ to some entry and subtract $\beta \geq \alpha$ from two "neighbor" entries and retain the monotonicity. First we obtain some formulas giving representations for these perturbations of A.

LEMMA 3. Let A be a triangular APM, $i \geq j+1$, and $\alpha \in \mathbf{R}$. Then $A + \alpha e_{i,j} = PA$, where $P = I + K$, and the entries of the matrix K $(= \alpha e_{i,j}A^{-1})$ are all zero except for the $i^{th}$ row which is of the form

$$\alpha(d^2(d-1)^{j-3} \quad d^2(d-1)^{j-4} \ \dots \ d^2(d-1) \quad d^2 \quad (d+1) \quad 1 \quad 0 \ \dots \ 0).$$

LEMMA 4. Assume $i \geq j+1$. Then

$$A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j} = PA$$

where

$$P = I - \beta e_{i-1,j}A^{-1} + \alpha e_{i,j}A^{-1} - \beta e_{i+1,j}A^{-1}.$$

Furthermore, if we let $P = I + K$, then the $i^{th}$ row of K is $\alpha$ times the vector

$$(d^2(d-1)^{j-3} \quad d^2(d-1)^{j-4} \ \dots \ d^2(d-1) \quad d^2 \quad (d+1) \quad 1 \quad 0 \ \dots \ 0)$$

and the $(i-1)$ and $(i+1)$ rows are $-\beta$ times the same vector. All other entries of K are zero.

LEMMA 5. For

$$M = ae_{i-1,j} + be_{i,j} + ce_{i+1,j}, \text{ where } i > j+1,$$

and for any lower triangular matrix T, $MTM = 0$.

LEMMA 6. For $i > j+1$ and $P = I + K$ as in Lemma 4, $P^{-1} = I-K$.

Proof: Note that $K = MA^{-1}$ where
$M = -\beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j}$, so $P = I + MA^{-1}$. Now
$(I+MA^{-1})(I-MA^{-1}) = I - MA^{-1}MA^{-1} = I - (MA^{-1}M)A^{-1} = I$,
since $MA^{-1}M = 0$ by Lemma 5. □

LEMMA 7. Let $i = j + 1$ and let P be as in Lemma 4. Then $P^{-1} = I - L$, where L is zero except that the $(i-1)$ and $(i+1)$ rows are $\frac{-\beta}{1-\beta}$ times the vector

$$(d^2(d-1)^{j-3} \quad d^2(d-1)^{j-4} \ \dots \ d^2 \quad (d+1) \quad 1 \quad 0 \ \dots \ 0)$$

and the $i^{th}$ row is $\frac{\alpha}{1-\beta}$ times the same vector.

Proof: From Lemma 4

$$P = I - \beta e_{i-1,j}A^{-1} + \alpha e_{i,j}A^{-1} - \beta e_{i+1,j}A^{-1}.$$

We will verify that

$$P^{-1} = I + \frac{\beta}{1-\beta} e_{i-1,j}A^{-1} - \frac{\alpha}{1-\beta} e_{i,j}A^{-1} + \frac{\beta}{1-\beta} e_{i+1,j}A^{-1}$$

by multiplying P with $P^{-1}$

First note that for any lower triangular matrix $T = (t_{ij})$,

$$e_{k,m}Te_{p,q} = t_{m,p}e_{k,q} = 0\,e_{k,q} = 0, \text{ if } m < p.$$

It follows then that the terms in the product $P\,P^{-1}$ of the form $c\,e_{k,j}A^{-1}e_{p,j}A^{-1}$ are equal to zero for $p = i$ or $i+1$, since $j = i - 1$. There are 6 such terms which are thus zero. Furthermore, the term $\frac{-\beta^2}{1-\beta}e_{i-1,j}A^{-1}e_{i-1,j}A^{-1} = \frac{-\beta^2}{1-\beta}e_{i-1,j}A^{-1}$, and when added together with the terms

$$\frac{\beta}{1-\beta}e_{i-1,j}A^{-1} - \beta e_{i-1,j}A^{-1},$$

the result is zero.

Similarly, $\frac{\alpha\beta}{1-\beta}e_{i,j}A^{-1}e_{i-1,j}A^{-1} = \frac{\alpha\beta}{1-\beta}e_{i,j}A^{-1}$ and together with the terms $\alpha e_{i,j}A^{-1} - \frac{\alpha}{1-\beta}e_{i,j}A^{-1}$, yields the zero matrix.

Finally, $\frac{-\beta^2}{1-\beta}e_{i+1,j}A^{-1}e_{i-1,j}A^{-1} = \frac{-\beta^2}{1-\beta}e_{i+1,j}A^{-1}$, and together with the terms $\frac{\beta}{1-\beta}e_{i+1,j}A^{-1} - \beta e_{i+1,j}A^{-1}$ yields the zero matrix. This establishes $PP^{-1} = I$.

THEOREM 2. Let A be a triangular APM with $d \geq 1$.

(a) If $i > j+1$ and $0 < \alpha \leq \beta$, then

$$(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} \geq 0.$$

(b) If $i = j + 1$ and $0 < \alpha \leq \beta < 1$, then

$$(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} \geq 0.$$

Proof: Assume $i > j+1$. From Lemma 4,

$$A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j} = PA, \text{ and so}$$

$$(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} = A^{-1}P^{-1}.$$

Recall that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ d+1 & 1 & 0 & \cdots & 0 \\ d^2 & d+1 & 1 & \cdots & 0 \\ d^2(d-1) & d^2 & d+1 & \cdots & \\ \vdots & \vdots & & & \vdots \\ d^2(d-1)^{n-3} & d^2(d-1)^{n-4} & \cdots & (d+1) & 1 \end{bmatrix}$$

and from Lemma 6, $P^{-1} = I - K$, where K is zero except for a 3xn submatrix in the i-1,i, and i+1 rows. Since K is of rank one, a typical column of P is the transpose of a vector of the form

$$(0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0 \ (\beta q) \ (-\alpha q) \ (\beta q) \ 0 \ \ldots \ 0),$$

where $q > 0$ and any of the strings of zeros may be of length 0. Since each entry of $A^{-1}P^{-1}$ is an inner product of a row of $A^{-1}$ and a column of $P^{-1}$, the expressions we need to consider to establish the nonnegativity of $A^{-1}P^{-1}$ are:

(1) $d^2(d-1)^k + \beta q d^2(d-1)^{j+2} - \alpha q d^2(d-1)^{j+1} + \beta q d^2(d-1)^j$, where $k > j + 2$ and $j \geq 0$;

(2) $d^2(d-1)^k + \beta q d^2(d-1) - \alpha q d^2 + \beta q(d+1)$, where $k > 1$;

(3) $d^2(d-1)^k + \beta q d^2 - \alpha q(d+1) + \beta q$, where $k \geq 1$;

(4) $d^2(d-1)^k + \beta q(d+1) - \alpha q$, where $k \geq 0$.

Expression (1) is nonnegative if

$$\beta(d-1)^2 - \alpha(d-1) + \beta \text{ is nonnegative.}$$

Clearly,

$$\beta(d-1)^2 - \alpha(d-1) + \beta \geq \beta(d-1)^2 - \beta(d-1) + \beta$$
$$= \beta((d-1)^2 - (d-1) + 1) \geq 0,$$

for all d.

Expression (2) is nonnegative if

$$\beta d^2(d-1) - \alpha d^2 + \beta(d+1) \text{ is nonnegative. Now}$$
$$\beta d^2(d-1) - \alpha d^2 + \beta(d+1)) \geq \beta d^2(d-1) - \beta d^2 + \beta(d+1)$$
$$= \beta(d(d-1)^2 + 1) \geq 0.$$

Expression (3) is nonnegative if $\beta d^2 - \alpha(d+1) + \beta$ is nonnegative. As before, we have

$\beta d^2 - \alpha(d+1) + \beta \geq \beta(d^2 - (d+1) + 1) = \beta(d^2 - d) \geq 0$, since $d \geq 1$.

Clearly, expression (4) is nonnegative.

Next consider the case where $i = j+1$. From Lemma 7, $P^{-1} = I - L = I - \frac{1}{1-\beta} K$. The discussion above for the case $i > j+1$ applies to all the entries of $A^{-1}P^{-1}$ except those in the

$j^{th}$ column. The $j^{th}$ column of $P^{-1}$ is the transpose of a vector of the form:

$$(0 \quad \cdots \quad 0 \quad \frac{1}{1-\beta} \quad \frac{-\alpha}{1-\beta} \quad \frac{\beta}{1-\beta} \quad 0 \quad \cdots \quad 0).$$

Again it is easily verified that the inner product of any row of $A^{-1}$ with this vector will always be nonnegative. □

By considering the special case $d = 1$, we obtain the following necessary and sufficient conditions for the nonnegativity of $(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1}$.

THEOREM 3. Let $A$ be a triangular APM with $d = 1$, and $\alpha, \beta$ be nonnegative numbers.

(a) For $i > j+1$,

$$(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} \geq 0$$

if and only if $\alpha \leq \beta$.

(b) For $i = j + 1$,

$$(A - \beta e_{i-1,j} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} \geq 0$$

if and only if $\beta < 1$, $1 + \beta \geq 2\alpha$ and $2\beta \geq \alpha$.

Proof: When $d = 1$,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ 2 & 1 & 0 & \cdots & & & 0 \\ 1 & 2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & 0 & & \cdots & & 2 & 1 \end{bmatrix}$$

Assume $i > j + 1$. The expressions (1), (2), (3), (4), in the proof of Theorem 2 reduce, when $d = 1$, to

(1') $\beta q$

(2') $-\alpha q + 2\beta q$

(3') $\beta q - 2\alpha q + \beta q$

(4') $2\beta q - \alpha q$.

It is easy to see then in order for all of these to be nonnegative, it is necessary that $\alpha \leq \beta$. Now suppose that

$i = j+1$. Then the entries of the $j^{th}$ column of $A^{-1}P^{-1}$ are inner products of the rows of $A^{-1}$ with the vector

$$(0 \ \ldots \ 0 \ \frac{1}{1-\beta} \ \frac{-\alpha}{1-\beta} \ \frac{\beta}{1-\beta} \ 0 \ \ldots \ 0).$$

The resulting expressions are:

(1") $\frac{1}{1-\beta}$

(2") $\frac{2}{1-\beta} + \frac{-\alpha}{1-\beta}$

(3") $\frac{1}{1-\beta} + \frac{-2\beta}{1-\beta} + \frac{\beta}{1-\beta}$

(4") $\frac{\alpha}{1-\beta} + \frac{2\beta}{1-\beta}$

Thus (1") implies $\beta < 1$; (3") implies $2\alpha \leq 1+\beta$; (4") implies $\alpha \leq 2\beta$. □

Theorems 2 and 3 have duals stated as below. These duals yield classes of monotone matrices which are obtained from an APM by perturbing the (i,j-1), (i,j) and (i,j+1) entries.

THEOREM 2*. Let A be a triangular APM with $d \geq 1$.

(a) If $i > j+1$ and $0 < \alpha \leq \beta$, then

$(A - \beta e_{i,j-1} + \alpha e_{i,j} - \beta e_{i,j+1})^{-1} \geq 0$

(b) If $i = j+1$ and $0 < \alpha \leq \beta < 1$, then

$(A - \beta e_{i,j-1} + \alpha e_{i,j} - \beta e_{i,j+1})^{-1} \geq 0$

THEOREM 3*. Let A be a triangular APM with $d = 1$, and $\alpha, \beta$ be nonnegative numbers.

(a) For $i > j+1$,

$(A - \beta e_{i,j-1} + \alpha e_{i,j} - \beta e_{i+1,j})^{-1} \geq 0$

if and only if $\alpha \leq \beta$.

(b) For $i = j+1$,

$(A - \beta e_{i,j-1} + \alpha e_{i,j} - \beta e_{i,j+1})^{-1} \geq 0$

if and only if $\beta < 1$, $1+\beta \geq 2\alpha$ and $2\beta \geq \alpha$.

REMARKS. (1) If $d > 1$ in Theorem 2 and Theorem $2^*$, then the condition $\alpha \leq \beta$ is not necessary.

(2) It is trivial that if A is a triangular APM with $d \geq 1$, then the matrices

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$$ have nonnegative Moore-Penrose inverses.

However, for $B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & -2 \end{bmatrix}$,

the Moore-Penrose inverse of B is

$$B^+ = \begin{bmatrix} 1/12 & -1/3 & -1/6 \\ 4/3 & -1/3 & -2/3 \end{bmatrix}.$$

## 4. SYMMETRIC ARITHMETIC PROGRESSION MATRICES

In this section we investigate the inverse of a symmetric APM.

LEMMA 8. If A is an nxn symmetric APM, then

$$A^{-1} = \begin{bmatrix} -a & -1/2d & 0 & \cdots & 0 & (-1)^{n-1}b \\ -1/2d & -1/d & -1/2d & 0 & \cdots & 0 \\ 0 & -1/2d & -1/d & -1/2d & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & & & & -1/2d & -1/d \\ (-1)^{n-1}b & 0 & \cdots & 0 & -1/2d & -a \end{bmatrix},$$

where $a = (d(n-2) + 2)/(2d(d(n-1) + 2))$ and $a + b = 1/2d$.

THEOREM 4. If $B = -A$ where A is a symmetric 2nx2n APM with $d \geq 1$, then

(i) $B^{-1} \geq 0$

(ii) B has (2n-1) positive eigenvalues and one negative eigenvalue.

Proof: Part (i) follows from the Lemma 8.

Let

$$C = 1/2d \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

and

$$W = \begin{bmatrix} -b & 0 & \cdots & 0 & b \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \\ 0 & 0 & \cdots & 0 & 0 \\ b & 0 & \cdots & 0 & -b \end{bmatrix} .$$

Then $B^{-1} = C + W$, where W is of rank one. Let $\lambda_1(C)$, $\lambda_2(C), \ldots, \lambda_{2n}(C)$ and $\lambda_1(B^{-1}), \lambda_2(B^{-1}), \ldots, \lambda_{2n}(B^{-1})$ denote the eigenvalues of C and $B^{-1}$ respectively, arranged in decreasing order. By Gerschgorin's Theorem, the eigenvalues of C are nonnegative. Elementary row operations can be used to establish that C has rank (2n-1). Thus $0 = \lambda_{2n}(C)$ and $0 < \lambda_{2n-1}(C)$. By the interlacing theorem ([2], Corollary 8.1-5), $\lambda_i(C+W) \in [\lambda_{i+1}(C), \lambda_i(C)]$, for $i = 1,2,\ldots,(2n-1)$. Hence $\lambda_{2n-1}(B^{-1}) > 0$ and so B and $B^{-1}$ have at least (2n-1) positive eigenvalues. However, $tr(B) < 0$, which implies B and $B^{-1}$ must have one negative eigenvalue.

REMARK. If A is any symmetric APM with $d \geq 1$, and $B = -A$, then B has exactly one negative eigenvalue.

We conclude this note by stating the following questions for further investigation of arithmetical progression matrices.

(1) Which classes of perturbed symmetric APM are monotone?

(2) Which classes of matrices have LU-decompositions such that L,U are respectively Toeplitz lower triangular, upper triangular APM. (This will, in turn, yield information on monotonicity of such matrices by Theorems 1 and 2.)

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## REFERENCES

1. A. Berman and R. L. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
2. G. H. Golub and C. F. Van Loan, Matrix Computations, The Johns Hopkins University, Baltimore, Maryland, 1983.
3. J. Lorenz and W. Mackens, Toeplitz matrices with totally nonnegative inverses, Linear Algebra and Appl. 24: 133-141 (1979).