Overdetermined Linear Systems Satisfying Nonnegativity Constraints

S. K. Jain*
Ohio University
Athens, Ohio 45701

Submitted by Robert J. Plemmons

ABSTRACT

Let A be a nonnegative $m \times n$ matrix, and let b be a nonnegative vector of dimension m. Also, let S be a subspace of $\mathbf{R}^n$ such that if P_S is the orthogonal projector onto S, then $P_S \geqslant 0$. A necessary condition is given for the matrix A to satisfy the following property: For all $b \geqslant 0$, if $\min\|b - Ax\|$ is attained at $x = x_0$, then $x_0 \geqslant 0$ and $x_0 \in S$. It is also shown that if a nonnegative matrix A has a nonnegative generalized inverse, then any submatrix of A also possesses a nonnegative generalized inverse.

1. DEFINITIONS AND NOTATION

Let A be an $m \times n$ real matrix, and let X be an $n \times m$ matrix. Consider the equations (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, (5) $AX = XA$. For any nonempty subset λ of $\{1,2,3,4,5\}$, X is called a λ-*inverse* of A, if X satisfies equations (i) for all $i \in \lambda$. If $\lambda = \{1,2,3,4\}$, then X is called the Moore-Penrose inverse of A, and is denoted by $A^{\dagger}$. A λ-inverse of A will be denoted by $A^{(\lambda)}$. Let S be a subspace of $\mathbb{R}^n$, and let P_S be the orthogonal projector onto S. Then an $n \times m$ matrix $X = P_S(AP_S)^{(\lambda)}$ is called a *S-restricted* λ-*inverse* of an $m \times n$ matrix A. If $A = (a_{ij})$ is such that $a_{ij} \geqslant 0$ for all i, j, we write $A \geqslant 0$, and we say that A is a nonnegative matrix.

2. INTRODUCTION AND PRELIMINARIES

Let A, b be nonnegative matrices of sizes $m \times n, m \times 1$ respectively, and let S be a subspace of $\mathbb{R}^n$. Approximate solutions of overdetermined linear systems $Ax = b$ which satisfy nonnegativity constraints $x \geqslant 0$, $x \in S$ are of

*This work was done while the author was visiting North Carolina State University at Raleigh in Fall 1981.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

considerable interest to several authors in view of their importance in problems in the applied sciences including economics (cf. [3], [5], [6]). The object of this paper is to find conditions on the matrix A such that for any $b \geqslant 0$, if $\min\|b - Ax\|$ is obtained at $x = x_0$, then $x_0 \geqslant 0$, $x_0 \in S$. Now the "constrained" linear system

$$Ax = b, \qquad x \in S, \quad x \geqslant 0, \tag{2.1}$$

is equivalent to the following unconstrained linear system:

$$\begin{bmatrix} A \\ P_{S^\perp} \end{bmatrix} x = \begin{bmatrix} b \\ 0 \end{bmatrix}, \tag{2.2}$$

where $P_{S^\perp}$ is the orthogonal projector onto the orthogonal complement $S^\perp$ of S. By regarding A as a linear operator from S to $\mathbb{R}^m$ the system (2.1) is also equivalent to

$$(AP_S)x = b, \qquad x \geqslant 0. \tag{2.3}$$

The best approximate solution of minimum norm of (2.2) and of (2.3) are respectively given by

$$x = \begin{bmatrix} A \\ P_{S^\perp} \end{bmatrix}^\dagger \begin{bmatrix} b \\ 0 \end{bmatrix}, \qquad x \geqslant 0, \tag{2.4}$$

and

$$x = P_S(AP_S)^\dagger b, \qquad x \geqslant 0. \tag{2.5}$$

Thus the study of nonnegative matrices A having nonnegative S-restricted Moore-Penrose inverse $P_S(AP_S)^\dagger$ is the same as finding conditions on A such that the system (2.1) has a nonnegative best approximate solution of minimum norm for all nonnegative vectors b. Theorem 1 gives a necessary condition for a nonnegative matrix A to have a nonnegative S-restricted Moore-Penrose inverse when the subspace S possesses a nonnegative orthonormal basis or equivalently $P_S \geqslant 0$ (cf [5, p. 65]). Theorem 1 also generalizes the well-known theorem of Plemmons and Cline [1, p. 123, Theorem 5.2]. The solution of the constrained linear system given in (2.4) suggests the following question: Does

$$\begin{bmatrix} A \\ B \end{bmatrix}^\dagger \geqslant 0 \quad \text{imply} \quad A^\dagger \geqslant 0, \quad B^\dagger \geqslant 0,$$

where $A \geqslant 0$, $B \geqslant 0$? Theorem 2 shows that the answer is yes.

3. MAIN RESULTS

We begin with a simple fact.

LEMMA. *Let S be a subspace of $\mathbb{R}^m$. Then $P_S \geq 0$ if and only if S possesses a nonnegative orthonormal basis.*

Proof. The "if" part follows from the fact that if $\{x_1, \dots, x_m\}$ is an orthonormal basis of S, then $P_S = x_1 x_1^T + \cdots + x_m x_m^T$. The converse is obvious. ∎

THEOREM 1. *Let A be a nonnegative $m \times n$ matrix and S be a subspace of $\mathbb{R}^m$ such that $P_S \geq 0$. Suppose $P_S(AP_S)^\dagger \geq 0$. Then A contains a submatrix A_{11} such that* $\operatorname{rank} A_{11} = \operatorname{rank}(AP_S)$ *and $A_{11}^\dagger \geq 0$, or more precisely, there exist permutation matrices P, Q of suitable orders such that*

$$PAQ = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where $T_{11}^\dagger \geq 0$, and $\operatorname{rank} T_{11} = \operatorname{rank}(AP_S)$.

Proof. Let

$$X = P_S(AP_S)^\dagger, \qquad Y = A.$$

We show that

$$\operatorname{rank}(XY) = \operatorname{rank}(YX) = \operatorname{rank} X. \tag{3.1}$$

Now

$$\operatorname{rank}(YX) = \operatorname{rank}(AP_S),$$

and

$$\operatorname{rank} X \leq \operatorname{rank}(AP_S)^\dagger = \operatorname{rank}(AP_S) = \operatorname{rank}(YX).$$

Hence $\operatorname{rank} X = \operatorname{rank} YX$. Further,

$$\operatorname{rank}(AP_S) \leq \operatorname{rank}\left[(AP_S)(AP_S)^\dagger A\right] \leq \operatorname{rank}\left[P_S(AP_S)^\dagger A\right] = \operatorname{rank}(XY).$$

Therefore

$$\operatorname{rank} X = \operatorname{rank}(YX) = \operatorname{rank}(AP_S) \leqslant \operatorname{rank}(XY) \leqslant \operatorname{rank} X,$$

which proves (3.1). Since XY, YX are nonnegative idempotents, each of rank r, say, and YX is symmetric, there exist permutation matrices P, Q such that

$$PXYP^T = \begin{bmatrix} K_1 & K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1K_1 & C_1K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad QYXQ^T = \begin{bmatrix} K_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.2)$$

where the diagonal blocks are square matrices; K_1, K_2 are each matrices of rank r; each of K_1, K_2 is a direct sum of positive rank one matrices; and C_1, D_1 are nonnegative matrices of suitable orders (some of the blocks may be absent).

Let $L = PXQ^T$, $M = QYP^T$, and partition L, M as $4\times 2, 2\times 4$ matrices so that the block multiplication of L and M in either order is possible. Let

$$L = (L_{ij}), \qquad 1 \leqslant i \leqslant 4, \quad 1 \leqslant j \leqslant 2.$$

$$M = (M_{kl}), \qquad 1 \leqslant k \leqslant 2, \quad 1 \leqslant l \leqslant 4.$$

From (3.2),

$$LM = \begin{bmatrix} K_1 & K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1K_1 & C_1K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad ML = \begin{bmatrix} K_2 & 0 \\ 0 & 0 \end{bmatrix},$$

and also (3.1) gives $\operatorname{rank}(LM) = \operatorname{rank}(ML) = \operatorname{rank} L = r$. Further, $\operatorname{diag} K_i > 0$, $i = 1, 2$. Then by equating block entries of LM and ML with the corresponding block entries obtained by actually multiplying $L = (L_{ij})$ and $M = (M_{kl})$, we obtain

$$L = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \\ L_{31} & L_{32} \\ 0 & L_{42} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 \\ 0 & M_{22} & M_{23} & M_{24} \end{bmatrix}$$

with

$$L_{11}M_{11}=K_1, \qquad M_{11}L_{11}=K_2.$$

Therefore,

$$\operatorname{rank}(L_{11}M_{11})=\operatorname{rank}(M_{11}L_{11})=\operatorname{rank}(L_{11})=r, \quad \text{and} \quad \operatorname{rank} M_{11}\geqslant r.$$

But then $L_{22}=0=L_{32}=L_{42}$. Hence we may rewrite L, M as 2×2 matrices with block entries as below:

$$L=\begin{bmatrix} L_{11} & 0 \\ L'_{21} & 0 \end{bmatrix}, \qquad M=\begin{bmatrix} M_{11} & M'_{12} \\ 0 & M'_{22} \end{bmatrix}, \tag{3.3}$$

where

$$L_{11}M_{11}=K_1=\begin{bmatrix} x_1y_1^T & & \\ & \ddots & \\ & & x_ry_r^T \end{bmatrix}, \tag{3.4}$$

$$M_{11}L_{11}=K_2=\begin{bmatrix} u_1u_1^T & & \\ & \ddots & \\ & & u_ru_r^T \end{bmatrix}, \tag{3.5}$$

where $x_i, y_i, u_i>0$ are unit vectors, and $y_i^Tx_i=1$, $1\leqslant i\leqslant r$.

Partition $L_{11}=(l_{ij})$, $M_{11}=(m_{ij})$ as $r\times r$ block matrices, so that the block multiplication of L_{11} with M_{11} in either order is possible and l_{ii}, m_{ii} respectively are of the same order as $x_iy_i^T, u_iu_i^T$. Then by performing actual multiplications $L_{11}M_{11}$ [and $M_{11}L_{11}$], and equating corresponding entries from the equation (3.4) [and (3.5)], we obtain a permutation σ of $\{1,2,\dots,r\}$ such that for each $i\in\{1,\dots,r\}$, $l_{i\sigma(i)}\neq 0$, $m_{\sigma(i)i}\neq 0$, and $l_{ij}=0=m_{ji}$ for all $j\neq\sigma(i)$, and $l_{i\sigma(i)}m_{\sigma(i)i}=x_iy_i^T$, $m_{\sigma(i)i}l_{i\sigma(i)}=u_{\sigma(i)}u_{\sigma(i)}^T$. Hence there exists a permutation matrix P_1 such that $P_1L_{11}P_1^T$ [and $P_1M_{11}P_1^T$] are direct sums of matrices of the types

(I) l_{ii} [and m_{ii}], where $\sigma(i)=i$, $i\in\{1,\dots,r\}$,

(II)

$$\begin{bmatrix} 0 & l_{i\sigma(i)} & 0 & \cdots & 0 \\ 0 & 0 & l_{\sigma(i)\sigma^2(i)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & l_{\sigma^{d-2}(i)\sigma^{d-1}(i)} \\ l_{\sigma^{d-1}(i)i} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.6}$$

where d is the smallest positive integer such that $\sigma^d(i)=i$, $i\in\{1,\dots,r\}$ (and a similar representation of summands of $P_1M_{11}P_1^T$).

Further, since rank $L_{11}=r$, it follows that each $l_{i\sigma(i)}$ is of rank 1. Also, $l_{i\sigma(i)}m_{\sigma(i)i}=x_iy_i^T$, $x_i>0$, $y_i>0$, implies that $l_{i\sigma(i)}$ has no zero rows and no zero columns. Thus $l_{i\sigma(i)}=a_ib_{\sigma(i)}^T$, $a_i>0$, $b_{\sigma(i)}>0$. Moreover, rank $M_{11}\geqslant r$ implies each $m_{\sigma(i)i}$ is of rank $\geqslant 1$, and likewise $l_{i\sigma(i)}$, $m_{\sigma(i)i}$ have no zero rows and no zero columns. We also note that the summands (say, S_1 and S_2) of types (I) and (II) of $P_1L_{11}P_1^T$ as exhibited in (3.6) have respectively nonnegative Moore-Penrose inverses, namely

$$\frac{1}{\|a_i\|\,\|b_i\|}S_1^T \quad\text{and}\quad DS_2^T,$$

where

$$D=\begin{bmatrix} \|a_i\|\,\|b_{\sigma(i)}\| & & \\ & \ddots & \\ & & \|a_{\sigma^{d-1}(i)}\|\,\|b_i\| \end{bmatrix}$$

is a diagonal matrix. Thus $L_{11}^{\dagger}\geqslant 0$. Since rank $L=$ rank L_{11}, we obtain from (3) that $L_{21}'=C'L_{11}$, for some matrix C'. But then $L_{21}'=C'L_{11}L_{11}^{\dagger}L_{11}=C'\,L_{11}$, $C''\geqslant 0$. Hence

$$L=\begin{bmatrix} L_{11} & 0 \\ C''L_{11} & 0 \end{bmatrix}, \qquad \text{where} \quad C''\geqslant 0.$$

Therefore by choosing

$$P_1'=\begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix}, \qquad P_2=P_1'P, \qquad Q_2=P_1'Q,$$

we have shown

$$P_2\left[(P_S)(AP_S)^{\dagger}\right]Q_2^T=\begin{bmatrix} S & 0\\ \Lambda S & 0\end{bmatrix},$$

$$Q_2AP_2^T=\begin{bmatrix} T_{11} & T_{12}\\ 0 & T_{13}\end{bmatrix},$$

where S and T_{11} are direct sums of matrices types (I) and (II), $\Lambda \geqslant 0$, and thus $S^{\dagger} \geqslant 0$ and $T_{11}^{\dagger} \geqslant 0$. This proves the theorem. ■

REMARK. If rank $A = \operatorname{rank} T_{11}$, then $T_{13}=0$, and $T_{12}=T_{11}D'$ for some matrix D' of suitable order. In this case A can be represented as

$$Q_2AP_2^T=\begin{bmatrix} T_{11} & T_{11}D'\\ 0 & 0\end{bmatrix}.$$

The following example shows that the condition obtained in the theorem is not sufficient.

EXAMPLE. Let

$$A=\begin{bmatrix} 1 & 0 & 0 & \alpha\\ 0 & 2 & 2 & \beta\\ 0 & 2 & 2 & \gamma\\ 0 & 0 & 0 & 1\end{bmatrix}, \qquad \text{where} \quad \alpha,\beta,\gamma \geqslant 0.$$

Let

$$P_S=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Then

$$AP_S=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\\ \sqrt{2} & \sqrt{2} & 2 & 0\\ \sqrt{2} & \sqrt{2} & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Now it is a straightforward verification to show that

$$(AP_S)^\dagger = P_S(AP_S)^\dagger.$$

We note that $P_S(AP_S)^\dagger$ is a $\{1,2,3\}$-inverse of AP_S. To show $P_S(AP_S)^\dagger$ is a $\{4\}$-inverse, we write $[P_S(AP_S)^\dagger](AP_S) = P_S[(AP_S)^\dagger(AP_S)]P_S$ and hence $P_S(AP_S)^\dagger(AP_S)$ is symmetric. But then $(AP_S)^\dagger \geqslant 0$ implies that there exists a monomial submatrix of rank equal to the rank of the matrix AP_S, which is 2 (cf. [4, Theorem 2]). Clearly AP_S does not possess a monomial submatrix of rank 2. Hence $P_S(AP_S)^\dagger$ is not nonnegative.

THEOREM 2. *Let A be a nonnegative matrix such that A has a nonnegative λ-inverse $A^{(\lambda)}$ where $1 \in \lambda$. Let A_1 be any submatrix of A. Then $A_1^{(\lambda)} \geqslant 0$. The converse holds also if* rank $A_1 =$ rank A.

The proof of the above theorem is an immediate consequence of the following lemmas.

LEMMA 1. *Let A be a nonnegative matrix and P, Q be permutation matrices of suitable sizes such that PAQ is defined. Then $(PAQ)^{(\lambda)} \geqslant 0$ if and only if $A^{(\lambda)} \geqslant 0$.*

Proof. Obvious. ■

LEMMA 2. *Let*

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \text{and} \quad 1 \in \lambda.$$

Then $A^{(\lambda)} \geqslant 0$ implies $A_i^{(\lambda)} \geqslant 0$, $i = 1,2$. A similar result holds if $A = [A_1 \ A_2]$.
Moreover, if rank $A =$ rank A_1, *then $A^{(\lambda)} \geqslant 0$ if and only if $A_1^{(\lambda)} \geqslant 0$.*

Proof. The proof follows from the characterization of nonnegative λ-monotone matrices [4, Theorem 1] by careful block partitioning of the matrices and straightforward computations. ■

The following example shows that the converse of Theorem 2 is not true in general.

EXAMPLE. Let

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \qquad \text{where} \quad A = [1 \quad 2], \quad A_2 = [3 \quad 4].$$

Then $A_1^\dagger \geqslant 0$ and $A_2^\dagger \geqslant 0$. But $A^{(\lambda)}$ is not nonnegative for any nonempty subset λ of $\{1,2,3,4,5\}$ containing 1.

The author would like to thank the referee for his comments and suggestions.

REFERENCES

1 A. Berman and R. L. Plemmons, *Nonnegative Matrices in Mathematical Sciences*, Academic, 1979.

2 P. Flor, On groups of nonnegative matrices, *Compositio Math.* 21:376–382 (1969).

3 K. H. Haskell and R. J. Hanson, An algorithm for linear least squares problems with equality and nonnegativity constraints, *Math. Programming* 21:98–118 (1981).

4 S. K. Jain and L. E. Snyder, Nonnegative λ-monotone matrices, *SIAM J. Algebra and Disc. Methods* 2:66–76 (1981).

5 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses — Theory and Applications*, Wiley, 1974.

6 B. Noble and J. W. Daniel, *Applied Linear Algebra*, Prentice-Hall, 1977.

Received 10 October 1982; revised 28 March 1983