Journal of the Indian Math. Soc.
46(1982) 85–89

NONNEGATIVE MINIMUM NORM LEAST SQUARES SOLUTIONS OF AX = B

By S.K. JAIN AND L.E. SNYDER

[Received November 16, 1981]

1. Notation and definitions. Let A denote an $m \times n$ matrix and X denote an $n \times m$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, and (5) $AX = XA$, where T denotes the transpose. Let λ denote a nonempty subset of $\{1, 2, 3, 4, 5\}$. Then X is called a λ-inverse of A if X satisfies equation (i) for each $i \in \lambda$. A λ-inverse of a matrix A is denoted by $A^{(\lambda)}$. A $\{1, 2, 3, 4\}$-inverse of A is the unique Moore-Penrose inverse of A and is denoted by $A^\dagger$. A $\{1, 2, 5\}$-inverse of A exists if and only if $m = n$ and rank A = rank A^2. A $\{1, 2, 5\}$-inverse is called a group inverse and is denoted by $A^\#$. The group inverse $A^\#$ is a polynomial in A. $A \geqslant 0$ means that all entries of A are nonnegative, and $R(A)$ denotes the range of A.

2. EXAMPLE. In this section we give an example for which the minimum norm least squares solution for the system $AX = I$ fails to be nonnegative, but for which the system $AX = B$ does have a nonnegative minimum norm least squares solution for some nonnegative idempotent matrix B.

Let

$$A = B = \begin{bmatrix} UU^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CUU^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } U = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

$C = (\ 1\ \ 2\ \ 3\ \ 4\ )$. It can be shown that $A^\dagger$, which is the minimum norm least squres solution of $AX = I$, is not nonnegative but that $A^\dagger A$, which is the minimum norm least squares solution of $AX = A$, is nonnegative. It is also of interest to note that $AA^\dagger \ngeq 0$ for this example.

3. RESULTS

LEMMA 1. *Let A be any square matrix and E be an idempotent matrix,*

(*i*) *If $R(A) \subset R(E)$, then $EA = A$.*

(*ii*) *If $AE = EA$ and rank A = rank (AE), then $R(A) \subset R(E)$ and so $A = EA$.*

PROOF The proof is straightforward.

LEMMA 2. *If A, B are square matrices such that rank (AB) = rank A, $BA = AB$, and $B^{\#}$ exists, then*

(*i*) $(A^{\dagger}BA)^{\#} = (A^{\dagger}BA)^{\dagger} = A^{\dagger}B^{\#}A$ *and*

(*ii*) $(AA^{\dagger}B)^{\#} = B^{\#2}AA^{\dagger}B.$

PROOF. It is a consequence of lemma 1 that $BB^{\#}A = A$. The verification of (i) and (ii) follows from direct computation using this fact together with $AB = BA$.

The theorem proved below gives the form of a nonnegative minimum norm least squares solution of $AX = B$, where A, B are nonnegative matrices satisfying certain conditions including $B^{\#} \geqslant 0$. The characterization of nonnegative matrices B for which $B^{\#} \geqslant 0$ is given in [4]. The theorem below generalizes the known result when $B = I$, [7].

Our theorem gives necessary conditions in order that a solution X_0 to the minimization problem:

$$\min \| AX-B \| ,$$

is also a solution of the constrained minimization problem.

$$\min \| AX-B \| , X \geqslant 0.$$

THEOREM 1. *Let A, B be nonnegative matrices such that $B^{\#} \geqslant 0$, $AB = BA$, and rank AB = rank A. If $AX = B$ has a nonnegative minimum norm least squares solution, then there are permutation matrices P, Q such that*

$$PAQ^T = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ZA_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $$QXP^T = \begin{bmatrix} X_{11} & X_{11}T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where some zero blockes may not appear and A_{11} X_{11}, are direct sums of matrices of the following two types (not necessarily both):

(*I*) $\beta x y^T$, $\beta > 0$, *x and y positive unit vectors*

(*II*) $$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & 0 \\ \beta_{d1}x_dy_1^T & 0 & \cdot & \cdot & & 0 \end{bmatrix}$$

$\beta_{ij} > 0$, x_i, y_i *positive unit vectors not necessarily of the same size.*

Equivalently $A_{11}^{\dagger} \geqslant 0$, $X_{11}^{\dagger} \geqslant 0$.

PROOF. It is known that the minimum norm least squares solution of $AX = B$ is given by $X_0 = A^{\dagger}B$, ([1], p. 119). By lemma 2, $(AX_0)^{\#} \geqslant 0$ and $(X_0A)^{\#} = (X_0A)^{\dagger} \geqslant 0$. Then by [3] and [4] there exist permutation matrices P_1, Q_1 such that

$$P_1AX_0P_1^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $$Q_1X_0AQ_1^T = \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where J and J_1 are direct sums of matrices of types (I) and (II) (not necessarily both) and C, $D \geqslant 0$.

Let $L = P_1AQ_1^T$, $M = Q_1X_0P_1^T$

Then

$$LM=\begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$ML=\begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the same argument as in Lemma 1 of [5], we obtain

$$L=\begin{bmatrix} L_{11} & L_{11}U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$M=\begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ VM_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where $L_{11}M_{11} = J$ and $M_{11}L_{11} = J_1$.

Arguments similar to the proof of lemma 2 in [6] can be used to obtain permutation matrices P_2, Q_2 such that

$$P_2L_{11}Q_2^T \text{ and } Q_2M_{11}P_2^T$$

are direct sums of matrices of types (I) or (II). The proof depends on the fact that each row of the block partitioned matrix has one and only one nonzero block, likewise for the columns. The details are rather techanical although straight forward.

Finally for

$$P=\begin{bmatrix} P_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$Q=\begin{bmatrix} Q_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

PAQ^T and QX_0P^T have the desired form. This completes the proof.

If the matrix B satisfies a somewhat weaker hypothesis than commut ing with A, then the following theorem gives the form of the matrix A. However in this case we are unable to yield the form of the solution.

THEOREM 2. *Let A, B be nonnegative matrices such that $B^{\#} \geqslant 0$, $R(A) \subset R(B)$, and rank A = rank AB. If the system $AX = B$ has a nonnegative minimum norm least squares solution, then there are permutation matrices P, Q such that*

$$PAQ = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where J is a direct sum of matrices of the types (I) and (II) as in Theorem 1.

REFERENCES

1. BEN-ISRAEL ADI and THOMAS N.E. GREVILLE, *Generalized Inverses: Theorey and Applications*, John Wiley & Sons, New Yoɪk, 1974.

2. BERMAN A. and R.L. PLEMMONS, *Nonnegative Matrices in The Mathematical Sciences*, Academic Press, 1979.

3. JAIN, S.K., V.K. GOEL and EDWARD K. KWAK, Nonnegative Matrices Having Some Nonnegative Moore-Penrose and Group Inverses, Linear and *Multilinear Algebra*, 7(1979), pp. 59–72.

4. ————, Decomposition of Nonnegative Group-monotone Matrices, *Trans. Amer. Math. Soc.*, 257 (1980), pp. 371–385.

5. JAIN, S.K. and L.E. SNYDER, Nonnegative λ-monotone Matrices, *SIAM J. Alg. Disc. Math.*, 2(1981), pp. 66–76.

6. JAIN, S.K. Nonnegative Matrices Having Nonnegative W-weighted Group Inverses, *Proc. Amer. Math. Soc.*, 85 (1982) pp 1–9.

7. PLEMMONS, R.L. and R.E. CLINE, The Generalized Inverses of a Nonnegative Matrix, *Proc. Amer. Math. Soc.*, 31(1972). pp. 46–50.

Department of Mathematics
Ohio University
Athens, OH 45701