Linear and Multilinear Algebra, 1984, Vol. 15, pp. 121–132
0308-1087/84/1502-0121 $18.50/0

Linear Systems Having Nonnegative Least Squares Solution

S. K. JAIN and L. E. SNYDER
Department of Mathematics, Ohio University, Athens, Ohio 45701

(*Received September 30, 1981; in final form January 17, 1983*)

Necessary conditions are developed for a system of the form $AX = B$ with $A > 0$ and $B > 0$ to have a least squares solution X which is nonnegative. Also it is shown that if a nonnegative matrix A has a nonnegative W-weighted $\{1,3\}$-inverse for some nonnegative positive definite symmetric matrix W, then A has a nonnegative $\{1,3\}$-inverse. As a consequence of this, a short proof is obtained of a recent theorem of Jain and Egawa concerning nonnegative best approximate solutions (*SIAM J. ALG. DISC. METH.*, 3 (1982), 197–213).

1. INTRODUCTION

This papers continues the study of finding conditions such that the system $AX = B$, $A \geqslant 0$, $B \geqslant 0$, has a least squares solution which is nonnegative. A large number of papers have previously obtained sufficient conditions by characterizing λ-monotone nonnegative matrices. A theorem of Berman–Plemmons [4, Theorem 5] gives necessary and sufficient conditions for the case where B is the identity matrix. If $B = B^2$, $AB = BA$, and rank $(AB) = \text{rank } A$, then a necessary condition that the system $AX = B$ has a least squares solution which is nonnegative was obtained by Egawa–Jain [5, Theorem 4.4]. The present paper generalizes further to the case when B has a nonnegative $\{1\}$-inverse, rank $(AB) = \text{rank } A$, and $R(A) \subset R(B)$. The proof of this theorem (Theorem 1) does not depend on the thoerem of Egawa–Jain, but their theorem is obtained as a consequence of Theorem 1. The concept of W-weighted generalized inverse has considerably simplified the computations. Theorem 2 gives a short proof of another theorem [5, Theorem 3.7] which states that if a nonnegative matrix A has a nonnegative W-weighted $\{1,3\}$-inverse, where W is a nonnegative matrix corresponding to a positive definite symmetric bilinear form, then A also possesses a nonnegative $\{1,3\}$-inverse.

2. NOTATION AND DEFINITIONS

All matrices considered are real.

$\mathbb{R}^m$: the vector space of $m \times 1$ matrices over the reals $\mathbb{R}$.

X^T: the transpose of the matrix X.

$R(B)$: the range of an $m \times n$ matrix B, i.e., $\{y \in \mathbb{R}^m \mid y = Bx$, for some $x \in \mathbb{R}^n\}$.

Let W denote a positive definite symmetric $n \times n$ matrix. The norm on $\mathbb{R}^n$ induced by W is defined by

$$\|x\|_W = \sqrt{x^T W x}\,, \qquad x \in \mathbb{R}^n.$$

Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then $x_0 \in \mathbb{R}^n$ is called a best approximate solution with respect to the norm $\|\cdot\|_W$ if $\|Ax_0 - b\|_W$ is minimum. If $W = I$, then $\|x\|_W$ is the usual euclidean norm of x and in this case a best approximate solution is commonly known as a least squares solution.

For X an $m \times n$ matrix, define a norm as follows:

$$\|X\|_2 = \sqrt{\operatorname{trace} X^T X}\,.$$

A matrix X is said to be a least squares solution of $AX = B$ if

$$\|AX - B\|_2 \qquad \text{is minimum.}$$

If A and X are $m \times n$ and $n \times m$ matrices, respectively, such that $AXA = A$, then X is called a $\{1\}$-inverse of A and is denoted by $A^{(1)}$. If X also satisfies $(AX)^T = AX$, then X is called a $\{1,3\}$-inverse of A and is denoted by $A^{(1,3)}$.

3. MAIN RESULTS

LEMMA 1 *If S is a nonnegative matrix such that $JS = S$, rank*

(SJ) = rank S, and if

$$J = \begin{bmatrix} x_1 y_1^T & 0 & \dots & 0 \\ 0 & x_2 y_2^T & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & x_r y_r^T \end{bmatrix}$$

where x_i, y_i are positive unit vectors with $y_i^T x_i = 1$, $1 \leqslant i \leqslant r$, then

$$S = \begin{bmatrix} \beta_{11} x_1 v_1^T & \beta_{12} x_1 v_2^T & \dots & \beta_{1r} x_1 v_r^T \\ \beta_{21} x_2 v_1^T & \beta_{22} x_2 v_2 T & \dots & \beta_{2r} x_2 v_r^T \\ \vdots & \vdots & & \\ \beta_{r1} x_r v_1^T & \beta_{r2} x_r v_2^T & \dots & \beta_{rr} x_r v_r^T \end{bmatrix}$$

where $\beta_{ij} \geqslant 0$ and the v_i's are nonzero nonnegative unit vectors.

Proof Let

$$S = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1r} \\ S_{21} & S_{22} & \dots & S_{2r} \\ \vdots & & & \\ S_{r1} & S_{r2} & \dots & S_{rr} \end{bmatrix}$$

be a block partitioning of S such that block multiplication with J is possible. Then

$$JS = \begin{bmatrix} x_1 y_1^T & & \\ & \ddots & \\ & & x_r y_r^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1r} \\ & \vdots & & \\ S_{r1} & S_{r2} & \dots & S_{rr} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 y_1^T S_{11} & x_1 y_1^T S_{12} & \dots & x_1 y_1^T S_{1r} \\ \vdots & & & \\ x_r y_r^T S_{r1} & x_r y_r^T S_{r2} & \dots & x_r y_r^T S_{rr} \end{bmatrix}$$

$$= S = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1r} \\ \vdots & & & \\ S_{r1} & S_{r2} & \dots & S_{rr} \end{bmatrix}$$

Thus $x_1 y_1^T S_{11} = S_{11}$, $x_1 y_1^T S_{12} = S_{12}$, and so on. It follows that for each $1 \leqslant i, j \leqslant r$, S_{ij} is of the form $\alpha_{ij} x_i v_{ij}^T$, where each v_{ij} is a nonzero nonnegative unit vector and $\alpha_{ij} \geqslant 0$.

Next, we show that each column block

$$[S]^j = \begin{bmatrix} S_{1j} \\ S_{2j} \\ \vdots \\ S_{rj} \end{bmatrix}$$

of S is also of rank 1. Let s be the rank of S, and suppose some column block is of rank 2 or more. Then there are $s - 1$ column blocks which generate the column space of S, i.e. any column block $[S]^j$ can be expressed as

$$[S]^j = [S]^{\alpha_1} D_1 + \cdots + [S]^{\alpha_{s-1}} D_{s-1}.$$

For notational convenience let us assume that $\alpha_1 = 1$, $\alpha_2 = 2$, $\ldots, \alpha_{s-1} = s - 1$. Then

$$\begin{aligned} \operatorname{rank}(SJ) &= \operatorname{rank}\left(([S]^1 [S]^2 \ldots [S]^r)J\right) \\ &= \operatorname{rank}\left([S]^1 J\, [S]^2 J \ldots [S]^r J\right) \\ &= \operatorname{rank}\left([S]^1 J\, [S]^2 J \ldots [S]^{s-1} J\right) \leqslant s - 1, \end{aligned}$$

since each $[S]^i J$ is of rank $\leqslant 1$. This contradicts the hypothesis that rank $(SJ) =$ rank $S = s$. It follows now that each block column can be expressed in the form

$$\begin{bmatrix} \beta_{1j} x_1 v_j^T \\ \vdots \\ \beta_{rj} x_r v_j^T \end{bmatrix}.$$

This completes the proof of the lemma.

Remark We can also write

$$S = U \mathscr{B} V^T$$

where

$$U = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & 0 & x_r \end{bmatrix}, \qquad B = (\beta_{ij}),$$

$$V^T = \begin{bmatrix} v_1^T & 0 & \dots & 0 \\ 0 & v_2^T & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & v_r^T \end{bmatrix}$$

Let B be a nonnegative matrix such that $B^{(1)} \geqslant 0$. (Such matrices have been characterized in [7]). Then $BB^{(1)}$ is a nonnegative idempotent matrix. By Flor [6], we may assume without loss of generality in the lemma which follows that

$$M = BB^{(1)} = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$J = \begin{bmatrix} x_1 y_1^T & & \\ & \ddots & \\ & & x_r y_r^T \end{bmatrix},$$

x_i, y_i are positive unit vectors such that $y_i^T x_i = 1$, $1 \leqslant i \leqslant r$, and C, D are nonnegative matrices of suitable sizes.

Lemma 2 *Let A, B be nonnegative square matrices such that $B^{(1)} \geqslant 0$. Then $R(A) \subset R(B)$, and rank $(AB) =$ rank A iff*

$$AE = \begin{bmatrix} K & KD_1 & KD_2 & KD_3 \\ 0 & 0 & 0 & 0 \\ CK & CKD_1 & CKD_2 & CKD_3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$E = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ C & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \qquad K = U\mathscr{B}V^T,$$

$$U = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & x_r \end{bmatrix}, \qquad V^T = \begin{bmatrix} v_1^T & 0 & \dots & 0 \\ 0 & v_2^T & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & v_r^T \end{bmatrix},$$

$\mathscr{B} = (\beta_{ij})$ is some $r \times r$ matrix, x_i's are positive unit vectors, v_i's are nonzero nonnegative unit vectors, and D_1, D_2, D_3 are matrices of suitable sizes. (Note that some of the zero blocks may be absent.)

Proof Set $M = BB^{(1)}$. Then $R(A) \subset R(M)$, $M = M^2$, and thus $A = MA$, rank $(AM) =$ rank A. As stated prior to Lemma 2, we shall take

$$M = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us partition $A = (X_{ij})$, $1 \leqslant i, j \leqslant 4$, in such a way that the block multiplication of M with A can be performed. Then it follows from $MA = A$ that

$$A = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ 0 & 0 & 0 & 0 \\ CX_{11} & CX_{12} & CX_{13} & CX_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $JX_{1j} = X_{1j}$, $1 \leqslant j \leqslant 4$. By computing AM, we get

$$\text{rank }(AM) = \text{rank }((X_{11} + X_{13}C)J).$$

Now

$$\begin{aligned} \text{rank }(X_{11} + X_{13}C)J &= \text{rank } A = \text{rank }(X_{11}\, X_{12}\, X_{13}\, X_{14}) \\ &= \text{rank }(X_{11} + X_{13}C\; X_{12}\, X_{13}\, X_{14}) \\ &\geqslant \text{rank }(X_{11} + X_{13}C). \end{aligned}$$

Hence

$$\text{rank } (X_{11} + X_{13}C)J = \text{rank } (X_{11} + X_{13}C) = \text{rank } AM = \text{rank } A.$$

For

$$E = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ C & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

$$AE = \begin{bmatrix} X_{11} + X_{13}C & X_{12} & X_{13} & X_{14} \\ 0 & 0 & 0 & 0 \\ CX_{11} + X_{13}C & CX_{12} & CX_{13} & CX_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} X_{11} + X_{13}C & (X_{11} + X_{13}C)D_1 & (X_{11} + X_{13}C)D_2 & (X_{11} + X_{13}C)D_3 \\ 0 & 0 & 0 & 0 \\ C(X_{11} + X_{13}C) & C(X_{11} + X_{13}C)D_1 & C(X_{11} + X_{13}C)D_2 & C(X_{11} + X_{13}C)D_3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

since rank A = rank $(X_{11} + X_{13}C)$. Now $J(X_{11} + X_{13}C) = X_{11} + X_{13}C$, and rank $(X_{11} + X_{13}C)J$ = rank AB = rank A = rank $(X_{11} + X_{13}C)$. Hence by Lemma 1

$$X_{11} + X_{13}C = \begin{bmatrix} \beta_{11}x_1v_1^T & \beta_{12}x_1v_2^T & \cdots \\ \beta_{21}x_2v_1^T & \cdots & \\ \vdots & \cdots & \vdots \end{bmatrix} = U\mathscr{B}V^T,$$

proving the "only if" part. The "if" part is a straightforward verification. This completes the proof.

With $BB^{(1)}$ in the form in Lemma 2, the matrix $\mathscr{B} = (\beta_{ij})$ shall be referred to as the coefficient matrix of A with respect to B.

LEMMA 3 *Let A, X be nonnegative matrices and let W be a nonnegative positive definite symmetric matrix such that*

$$AXA = A,$$
$$(WAX)^T = WAX,$$

i.e. A has a nonnegative W-weighted $\{1,3\}$-inverse. Then A has a nonnegative $\{1,3\}$-inverse.

Proof $(WAX)^T = WAX \Rightarrow (AX)^T W = W(AX) \Rightarrow$ if the ith column of AX is zero, then the ith row of AX is also zero. Since AX and XA are idempotents, it follows from [6] that there exist permutation matrices P, Q such that

$$PAXP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$QXAQ^T = \begin{bmatrix} J' & J'D' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C'J' & C'J'D' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J, J', C', D', D have the usual properties when a nonnegative idempotent is represented in the Flor's form [6].

Set $L = PAQ^T$, $M = QXP^T$, and proceed as in the proof of Lemma 1 in [7]. It follows that

$$L = \begin{bmatrix} L_{11} & L_{11}Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{11}Z' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X'M_{11} & X'M_{11}Z' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some matrices Z, Z', X' (not necessarily nonnegative) where

$$L_{11}M_{11} = J, \qquad M_{11}L_{11} = J'.$$

Thus by Lemma 2 in [7], L_{11} has a nonnegative $\{1\}$-inverse $L_{11}^{(1)}$, and so $L_{11}Z = L_{11}L_{11}^{(1)}L_{11}Z = L_{11}Z_0$, $Z_0 \geqslant 0$.

$$PAQ^T = \begin{bmatrix} L_{11} & L_{11}Z_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also, by Lemma 2 in [7], it follows (though not stated explicitly there) that $L_{11}^{(1,3)} \geqslant 0$. This implies that A has a nonnegative $\{1,3\}$-inverse, namely

$$A^{(1,3)} = Q^T \begin{bmatrix} L_{11}^{(1,3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$

This completes the proof of the lemma.

In the following theorem V^T and E are matrices as in Lemma 2, and V'^T denotes the matrix $(V^T \; V^TD_1 \; V^TD_2 \; V^TD_3)$ where D_1, D_2, D_3

are matrices of suitable sizes. We now prove

THEOREM 1 *Let A, B be nonnegative matrices such that B possesses a nonnegative $\{1\}$-inverse $B^{(1)}$, $R(A) \subset R(B)$, and rank $(AB) =$ rank A. If the system $AX = B$ has a least squares solution which is nonnegative and if $V'^T E^{-1} \geqslant 0$, then the coefficient matrix $\mathscr{B} = (\beta_{ij})$ of A with respect to B is of the following form*

$$P\mathscr{B}Q^T = \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix} QE \begin{bmatrix} V \\ 0 \end{bmatrix} Q^T,$$

where P, Q are permutation matrices of suitable sizes, and J' is a direct sum of matrices of following two types (not necessarily both):

(I) $\mu a b^T$, $\mu > 0$, *a, b are positive unit vectors.*

$$\text{(II)} \quad \begin{bmatrix} 0 & \mu_{12} a_1 b_2^T & 0 & \dots & 0 \\ 0 & 0 & \mu_{23} a_2 b_3^T & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & \mu_{r-1} a_{r-1} b_r^T \\ \mu_{r,1} a_r b_1^T & 0 & 0 & \dots & 0 \end{bmatrix},$$

$\mu_{ij} > 0$, *a_i, b_i are positive unit vectors, not necessarily of the same sizes, and $D' \geqslant 0$.*

Proof By Lemma 2

$$AE = \begin{bmatrix} U\mathscr{B}V^T & U\mathscr{B}V^T D_1 & U\mathscr{B}V^T D_2 & U\mathscr{B}V^T D_3 \\ 0 & 0 & 0 & 0 \\ CU\mathscr{B}V^T & CU\mathscr{B}V^T D_1 & CU\mathscr{B}V^T D_2 & CU\mathscr{B}V^T D_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} U \\ 0 \\ CU \\ 0 \end{bmatrix} \mathscr{B}(V^T \; V^T D_1 \; V^T D_2 \; V^T D_3) = U'\mathscr{B}V'^T, \text{ say.}$$

Also, in the notation of Lemma 2,

$$M = BB^{(1)} = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U'V_0'^T,$$

where $V_0'^T = (V_0^T \; V_0^T D \; 0 \; 0)$, and

$$V_0^T = \begin{bmatrix} y_1^T & & & \\ & y_2^T & & \\ & & \ddots & \\ & & & y_r^T \end{bmatrix}$$

We now show that the following nine statements are equivalent. Let $\mathscr{C} = \mathscr{B} V'^T E^{-1}$.

(1) $AX = B$ has a least squares solution which is nonnegative.

(2) $A^T AX = A^T B$ has a nonnegative solution.

(3) $A^T AX = A^T BB^{(1)}$ has a nonnegative solution.

(4) $E^{-1T} V' \mathscr{B}^T U'^T U' \mathscr{B} V'^T E^{-1} X = E^{-1} V' \mathscr{B}^T U'^T U' V_0'^T$ has a nonnegative solution.

(5) $\mathscr{C}^T W \mathscr{C} X = \mathscr{C}^T W V_0'^T$ has a nonnegative solution, where $W = U'^T U'$.

(6) $(\mathscr{C}^T \sqrt{W})(\sqrt{W}\,\mathscr{C}) Y = (\mathscr{C}^T \sqrt{W})\sqrt{W}$ has a nonnegative solution.

(7) $(\sqrt{W}\,\mathscr{C}) Y = \sqrt{W}$ has a least squares solution which is nonnegative.

(8) $(\sqrt{W}\,\mathscr{C}) Y = (\sqrt{W}\,\mathscr{C})(\sqrt{W}\,\mathscr{C})^{(1,3)}\sqrt{W}$ has a nonnegative solution.

(9) There is a nonnegative Y such that $\mathscr{C} Y \mathscr{C} = \mathscr{C}$ and $(W \mathscr{C} Y)^T = W \mathscr{C} Y$.

(1)$\Leftrightarrow$(2) and (6)$\Leftrightarrow$(7)$\Leftrightarrow$(8) are consequences of well known theorems.

(3)$\Leftrightarrow$(4) follows by substitution.

(5)$\Leftrightarrow$(6) depends on the existence of a nonnegative right inverse of $V_0'^T$, namely

$$\begin{bmatrix} V_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To show that (8)$\Rightarrow$(9), multiply the equation $(\sqrt{W}\,\mathscr{C}) Y = \sqrt{W}\,\mathscr{C}(\sqrt{W}\,\mathscr{C})^{(1,3)}\sqrt{W}$ on the right by $\mathscr{C}$ to obtain $\sqrt{W}\,\mathscr{C} Y \mathscr{C} = \sqrt{W}\,\mathscr{C}$, and then multiply on the left by the inverse of $\sqrt{W}$. Next multiply the equation in (8) on the left by $\sqrt{W}$ to obtain $W \mathscr{C} Y = \sqrt{W}(\sqrt{W}\,\mathscr{C})(\sqrt{W}\,\mathscr{C})^{(1,3)}\sqrt{W}$. The right side is symmetric and so the left side is symmetric also. Now assume (9) holds. Then we have $(\sqrt{W}\mathscr{C})(YW^{-1/2})(\sqrt{W}\mathscr{C}) = \sqrt{W}\mathscr{C}$. This implies $YW^{-1/2}$ is a $\{1\}$-inverse of $\sqrt{W}\mathscr{C}$. Next we show it is also a $\{3\}$-inverse. Consider:

$$\begin{aligned}(\sqrt{W}\mathscr{C}YW^{-1/2})^T &= (W^{-1/2}(W\mathscr{C}Y)W^{-1/2})^T\\ &= W^{-1/2}(W\mathscr{C}Y)^T W^{-1/2}\\ &= W^{-1/2}W\mathscr{C}YW^{-1/2}\\ &= \sqrt{W}\mathscr{C}YW^{-1/2}.\end{aligned}$$

Any $\{1,3\}$-inverse Z of $\sqrt{W}\mathscr{C}$ satisfies $(\sqrt{W}\mathscr{C})Z = \sqrt{W}\mathscr{C}(\sqrt{W}\mathscr{C})^{(1,3)}$. For $Z = YW^{-1/2}$ we have $\sqrt{W}\mathscr{C}YW^{-1/2} = \sqrt{W}\mathscr{C}(\sqrt{W}\mathscr{C})^{(1,3)}$ which gives us (8). Thus statement (9) and Lemma 3 imply that $\mathscr{C}$ has a nonnegative $\{1,3\}$-inverse.

Hence there exists permutation matrices P, Q such that

$$P\mathscr{C}Q^T = \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix},$$

where J' is a direct sum of matrices of types (I) and (II) (not necessarily both), $D' \geqslant 0$. Then

$$P\mathscr{B}V'^TE^{-1}Q^T = \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix},$$

yields

$$\begin{aligned}P\mathscr{B}Q^T &= \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix}\cdot\left(QE(V'^T)_R Q^T\right)\\ &= \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix}\left(QE\begin{bmatrix} V \\ 0 \end{bmatrix}Q^T\right)\\ &= \begin{bmatrix} J' & J'D' \\ 0 & 0 \end{bmatrix}QE\begin{bmatrix} V \\ 0 \end{bmatrix}Q^T,\end{aligned}$$

completing the proof.

Remark With the notation of the above theorem, if B is a nonnegative idempotent matrix such that $AB = BA$, then $V'^TE^{-1} \geqslant 0$. We show that $\mathscr{B}$ has indeed $\{1,3\}$-inverse. Now from (1) and (2), after cancellation and groupings, we obtain that X is a least squares solution of $AX = B$ if and only if

$$\mathscr{B}^TW\mathscr{B}Z = \mathscr{B}^TW, \qquad \text{where} \quad Z = V'^TE^{-1}X(V_0'^T)_R \geqslant 0.$$

Then as in the proof of the theorem we obtain that $\mathscr{B}$ has a nonnegative W-weighted $\{1,3\}$-inverse, and hence $\mathscr{B}$ has a nonnegative $\{1,3\}$-inverse (Lemma 3). Thus one obtains the theorem of Egawa and Jain [5, Thoerem 4.4] as a consequence of our theorem.

The concept of W-weighted generalized inverse also enables us to give a nice short proof of the following.

THEOREM 2 [5, Theorem 3.7] *Let A be an $m \times n$ nonnegative matrix. Let W be a positive definite symmetric bilinear form over $\mathbb{R}^m$ whose associated matrix with respect to standard basis is nonnegative. Suppose $Ax = b$ has a nonnegative best approximate solution with respect to W for all $b \geqslant 0$. Then $Ax = b$ has a nonnegative best approximate solution with respect to the euclidean norm.*

Proof It is known that $\|Ax - b\|_w$ is minimized if and only if $A^TWAx = A^TWb$. Together with this it follows from the hypotheses of the theorem that there exists a nonnegative matrix X such that $A^TWAX = A^TW$. Then an argument just like the proof of Theorem 1 yields $AXA = A$ and $(WAX^T = WAX$, and hence by Lemma 3 there exists a nonnegative $\{1,3\}$-inverse of A. Consequently for each $b \geqslant 0$, $Ax = b$ has a nonnegative least squares solution.

Acknowledgments

The authors would like to thank the referees for helpful suggestions and comments.

References

[1] Adi Ben-Israel and Thomas N. E. Greville, *Generalized Inverses: Theory and Applications*, John Wiley & Sons, New York, 1974.
[2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
[3] ———, *SIAM J. Appl. Math.* **22** (1972), 155–161.
[4] ———, Inverses of nonnegative matrices, *Linear and Multilinear Algebra* **2** (1974), 161–172.
[5] Y. Egawa and S. K. Jain, Nonnegative matrices A such that $Ax = b$ has nonnegative best approximate solution, *SIAM J. Alg. Disc. Math.* **3** (1982), 197–213.
[6] P. Flor, On groups of nonnegative matrices, *Compositio Math.* **21** (1969), 376–382.
[7] S. K. Jain and L. E. Snyder, Nonnegative λ-monotone matrices, *SIAM J. Alg. Disc. Math.* **2** (1981), 66–76.