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NONNEGATIVE MATRICES A SUCH THAT $Ax=b$ HAS NONNEGATIVE BEST APPROXIMATE SOLUTION*

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Abstract. In this paper the question of existence of nonnegative best approximate solutions (b.a.s.) of the linear system $Ax = b$ is investigated. Firstly, a necessary condition that $Ax = b$ have a nonnegative b.a.s. for all $b \geqq 0$ with respect to a positive definite symmetric bilinear form S whose associated matrix is nonnegative is obtained. It follows as a consequence that $Ax = b$ also has a nonnegative least squares solution (l.s.s.). Among other results it is proved that if B is a nonnegative idempotent matrix such that $AB = BA$, rank $(AB) =$ rank A, then $Ax = b$ has a nonnegative l.s.s. for all $b \in R(B)$, $b \geqq 0$, if and only if for certain well-defined matrix A_0 (called the coefficient matrix of A with respect to B) and certain symmetric bilinear form S, $A_0x = b$ has a nonnegative b.a.s. with respect to S. These results generalize the well-known results concerning the question of the existence of a nonnegative l.s.s. for the system $Ax = b$. Indeed, these investigations initiate a new approach to the question beyond the technique of inverse-positivity. The importance of this question lies in its varied applications to problems in mathematical economics, in probability theory, in operations research and in numerical algebra.

1. Notation and definitions.

$\mathbb{R}^m$: the vector space of $m \times 1$ matrices over the reals $\mathbb{R}$.
$\|x\|_2$: the usual Euclidean norm of a vector x.
$(X)^i$: the ith column of a matrix X.
$(X)_i$: the ith row of a matrix X.
$(x)_i$: the ith entry of a vector x.
$X_{i,j}$: the (i, j)th entry of a matrix X; thus, $((X)^j)_i = X_{i,j}$.
X^t: the transpose of a matrix X.
$R(B)$: the range of an $m \times n$ matrix B, i.e., $\{y \in \mathbb{R}^m | y = Bx$, for some $x \in \mathbb{R}^n\}$.
$R(B)^{\perp s}$: the subspace of $\mathbb{R}^m$ consisting of vectors x such that $S(x, b) = 0$ for all $b \in R(B)$, where S is a positive definite symmetric bilinear form.
$R(B)^{\perp}$: the subspace $R(B)^{\perp s}$ when S is the usual inner product on $\mathbb{R}^m$.
$\langle Y \rangle$: the subspace spanned by the subset Y of $\mathbb{R}^m$.
e_i: the vector in $\mathbb{R}^m$ having all entries zero except the ith entry, which is 1.
0_m: the zero vector in $\mathbb{R}^m$.

Let $S(\cdot, \cdot)$ be a positive definite symmetric bilinear form over $\mathbb{R}^m$. Then the associated symmetric matrix with respect to the standard basis $(e_1, e_2, \ldots, e_m)$, shall also be denoted by S, i.e., $S(x, y) = x^tSy$, $x, y \in \mathbb{R}^m$.

Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. Then $x_0 \in \mathbb{R}^n$ is called a best approximate solution of the system $Ax = b$ with respect to (w.r.t.) S if $S(Ax_0 - b, Ax_0 - b)$ is minimum. If S is the usual Euclidean norm, then the best approximate solution with respect to S is commonly known as the least squares solution.

If A, X are respectively $m \times n$, $n \times m$ matrices such that $AXA = A$ and $(AX)^t = AX$, then X is called a $\{1, 3\}$-inverse of A and is denoted by $A^{(1,3)}$.

For simplicity, we shall not indicate the order of matrices if it is clear from the context. Further, all matrices are real.

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2. Introduction. This paper addresses the question of characterizing nonnegative matrices A such that the linear system $Ax = b$, for certain nonnegative vectors b, has a nonnegative best approximate solution. The importance of this question can hardly be overemphasized in view of the fact that in many of the applications of nonnegative matrices one is involved in finding nonnegative solutions or least squares solutions of the system $Ax = b$, where $A \geqq 0$, $b \geqq 0$. For example, one finds numerous applications in areas such as mathematical economics, probability theory, numerical algebra and linear programming.

Since the nonnegativity of certain generalized inverses of A is related to the existence of nonnegative least squares solution of the system $Ax = b$, many authors have previously considered this question from this viewpoint. For example, the existence of a nonnegative $\{1, 3\}$-inverse of A is equivalent to the existence of a nonnegative least squares solution of $Ax = b$ for all nonnegative vectors b. The characterization of nonnegative matrices A having a certain nonnegative generalized inverse has been extensively studied in the literature (see [1]–[3], [6]–[11]).

We begin by considering the nonnegative best approximate solutions of $Ax = b$ with respect to an arbitrary positive definite symmetric bilinear form S whose associated matrix is nonnegative, and show in Theorem 3.7 that $A^{(1,3)} \geqq 0$—a well-known result for the Euclidean norm. We then proceed to the main question addressed in this paper, that of characterizing nonnegative matrices A such that $Ax = b$ has a nonnegative least squares solution for all nonnegative vectors b in a given set. We study this question in the case when $b \in R(B)$, where B is a nonnegative idempotent matrix such that $AB = BA$, rank $(AB) =$ rank A (Theorem 4.4). This is done by first obtaining the characterization of nonnegative matrices A which commute with a given nonnegative idempotent matrix B such that rank $(AB) =$ rank A (Lemma 4.2). We then introduce an intrinsic matrix A_0 (coefficient matrix) of A. The problem of the nonnegative least squares solution of $Ax = b$, $b \in R(B)$, $b \geqq 0$, is then reduced to the problem of obtaining a nonnegative best approximate solution of $A_0x = b$, for all nonnegative vectors b, with respect to some suitably defined norm S (Theorem 4.3). The proof of Theorem 4.4 is then completed by applying Theorem 3.7 and Theorem 4.3. An example is given to show that the converse of Theorem 4.4 does not, in general, hold.

We emphasize that Theorem 4.4 is an initial attempt to study the question stated in the beginning of the introduction. That this theorem is also true under a certain weaker hypothesis is explained in Remark 3 at the end of the paper. However, it is desirable that the hypothesis in Theorem 4.4 be further weakened. This remains open.

We remark that Lemmas 3.1, 3.6 and 4.2 are also of independent interest.

3. Nonnegative best approximate solutions.

LEMMA 3.1. *Let A be a nonnegative $m \times n$ matrix of rank r. Suppose $Ax = b$ has a nonnegative solution for every $b \geqq 0$, which makes this equation consistent. Then there exist permutation matrices P, Q such that*

$$(PAQ)_{i,i} \neq 0, \qquad 1 \leqq i \leqq r,$$

$$(PAQ)_{i,j} = 0, \qquad 1 \leqq i < j \leqq r.$$

Proof. We proceed by induction on r. Let $\mathcal{Q}$ denote the set of ordered pairs (P, Q) of permutation matrices such that $(PAQ)^j$, $1 \leqq j \leqq r$, are linearly independent, and $(PAQ)_{1,1} \neq 0$. For each $(P, Q) \in \mathcal{Q}$, we define $q(P, Q)$ as follows:

$$q(P, Q) = \text{card}\,\{j \mid 1 \leqq j \leqq r, (PAQ)_{1,j} \neq 0\}.$$

Let

$$p = \min\{q(P, Q) | (P, Q) \in \mathscr{Q}\}.$$

We want to prove $p = 1$. Suppose $p \geqq 2$. Let

$$\mathscr{F} = \{(P, Q) \in \mathscr{Q} | q(P, Q) = p, (PAQ)_{1,2} \neq 0\}.$$

Let

$$\alpha = \min\{(PAQ)_{1,1}/(PAQ)_{1,2} | (P, Q) \in \mathscr{F}\}.$$

Suppose (P, Q) is an element of $\mathscr{F}$ which gives this minimum value α. Set $F = PAQ$. Let

$$L = \{j | 1 \leqq j \leqq r, F_{1,j} = 0\},$$

$$K = \{i | 1 \leqq i \leqq m, \alpha F_{i,2} > F_{i,1}, F_{i,j} = 0 \ \forall j \in L\}.$$

Suppose $K \neq \varnothing$ and $i \in K$. Then, if we replace the first row by the ith row, this contradicts the minimality of α unless $F_{i,j} = 0$ for some $j \notin L$, $1 \leqq j \leqq r$. In the latter case, we get a contradiction to the minimality of p. Therefore $K = \varnothing$. This implies there exist nonnegative numbers β_j's, $j \in L$ such that

$$b = (F)^1 - \alpha(F)^2 + \sum_{j \in L} \beta_j (F)^j \geqq 0.$$

By our assumption the system $Fx = b$ has a nonnegative solution. Thus there exist $\gamma_j \geqq 0$, $1 \leqq j \leqq n$, such that $b = \sum_{j=1}^{n} \gamma_j (F)^j$. By the replacement theorem, we can choose k with $\gamma_k \neq 0$ such that

$$\langle (F)^j | 1 \leqq j \leqq r \rangle = \langle (F)^k, (F)^j | 1 \leqq j \leqq r, j \neq 2 \rangle.$$

Since $(b)_1 = 0$ and $\gamma_k \neq 0$, we have $F_{1,k} = 0$. Hence, if we replace the second column by the kth column, we get a contradiction to the minimality of p. Thus $p = 1$. By interchanging rows and columns suitably, we may assume that there exists $l \geqq r$ such that

$$A_{1,j} = 0, \quad 2 \leqq j \leqq l, \qquad A_{1,j} \neq 0, \quad j \geqq l+1 \text{ and } j = 1,$$

and the submatrix A' of A which consists of the columns 2 through l of A is of rank $r-1$. One can check that A' satisfies the hypothesis of the lemma. Consider the submatrix A'_0 consisting of all but the first row of A'. Since the first row of A' is a zero vector, we may assume by applying induction to A'_0 that

$$A_{i,j} = 0, \quad 2 \leqq i < j \leqq r, \qquad A_{i,i} \neq 0, \quad 2 \leqq i \leqq r.$$

Since $A_{1,j} = 0$, $2 \leqq j \leqq r$, and $A_{1,1} \neq 0$, the proof is complete.

In our next lemma, we shall need the following notation. Let $u \in \mathbb{R}^m$. We define

$$Z(u) = \{i | 1 \leqq i \leqq m, (u)_i \neq 0\},$$

$$Z_1(u) = \{i \in Z(u) | 1 \leqq i \leqq r\}.$$

LEMMA 3.2. *Let A be an $m \times n$ nonnegative matrix of rank r. Suppose that for every integer l_1, $1 \leqq l_1 \leqq n$, and for every subset L of $\{1, 2, \cdots, n\}$ with $Z_1((A)^{l_1}) \subseteq Z_1(\sum_{l \in L} (A)^l)$ we have the inclusion $Z((A)^{l_1}) \subseteq Z(\sum_{l \in L} (A)^l)$. (If L is empty, then by $\sum_{l \in L} (A)^l$ we understand the zero vector.) Also suppose that $Ax = b$ has a nonnegative solution for every $b \geqq 0$ which makes this equation consistent. Then there exist permutation* *matrices P, Q such that*

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix},$$

where P_1 is a permutation matrix of order r and

$$(PAQ)_{i,i} \neq 0, \qquad 1 \leqq i \leqq r,$$

$$(PAQ)_{i,j} = 0, \qquad 1 \leqq i \leqq r, \quad 1 \leqq j \leqq r, \quad i \neq j.$$

Proof. Let P', Q' be permutation matrices which satisfy the conclusion of Lemma 3.1. Now, since $(P'AQ')_{j,j} \neq 0$ and $(P'AQ')_{j,l} = 0$ for $j < l \leqq r$, we have

$$Z\left(\sum_{l=j+1}^{r} (P'AQ')^l\right) \not\supseteq Z((P'AQ')^j), \qquad 1 \leqq j \leqq r.$$

Therefore,

$$Z\left(\sum_{l=j+1}^{r} (AQ')^l\right) \not\supseteq Z((AQ')^j), \qquad 1 \leqq j \leqq r.$$

But then by assumption

$$Z_1\left(\sum_{l=j+1}^{r} (AQ')^l\right) \not\supseteq Z_1((AQ')^j),$$

and so by choosing $j = r, r-1, \ldots, 1$, we obtain

$$1 \leqq \operatorname{card}(Z_1((AQ')^r)) \neq \operatorname{card}\left(Z_1\left(\sum_{l=r-1}^{r} (AQ')^l\right)\right)$$

$$\neq \cdots \neq \operatorname{card}\left(Z_1\left(\sum_{l=1}^{r} (AQ')^l\right)\right) \leqq r.$$

Hence there exists a permutation matrix P satisfying the following property:

(*) $\qquad (PAQ')_{i,i} \neq 0, \quad 1 \leqq i \leqq r \qquad (PAQ')_{i,j} = 0, \quad 1 \leqq i < j \leqq r,$

where

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix};$$

P_1 is a permutation matrix of order r. With the matrix P as obtained above, let

$$\mathcal{B} = \{Q, \text{ a permutation matrix} \mid PAQ \text{ has the property } (*)\}.$$

By way of contradiction, suppose Lemma 3.2 is false. For each $Q \in \mathcal{B}$, let $\lambda(Q)$ be the positive integer defined as follows:

$$\lambda(Q) = \max\{i \mid 2 \leqq i \leqq r \text{ such that } \exists j,\ 1 \leqq j \leqq i-1, \text{ with } (PAQ)_{i,j} \neq 0\}.$$

Let

$$q = \min\{\lambda(Q) \mid Q \in \mathcal{B}\}, \qquad \mathcal{C} = \{Q \in \mathcal{B} \mid \lambda(Q) = q\}.$$

Let

$$\alpha = \min\left\{\sum_{1 \leqq j \leqq q-1} \frac{(PAQ)_{q,j}}{(PAQ)_{j,j}} \,\middle|\, Q \in \mathcal{C}\right\}.$$

Now let Q be an element of $\mathscr{C}$ which gives the minimum value α. Set $F = PAQ$. We know there exists j_0, $1 \leqq j_0 \leqq q-1$, such that $F_{q,j_0} \neq 0$. By the maximality of $\lambda(Q)$,

$$F_{i,q} = 0, \qquad i \neq q, \quad 1 \leqq i \leqq r.$$

Therefore $Z_1((F)^q) \subseteq Z_1((F)^{j_0})$. Since F satisfies the hypothesis of Lemma 3.2, $Z((F)^q) \subseteq Z((F)^{j_0})$. Therefore, there exists $\beta > 0$ such that $(F)^{j_0} - \beta(F)^q \geqq 0$. Let us set

$$(1) \qquad f = (F)^{j_0} - \beta(F)^q.$$

Then $Fx = f$ must have a nonnegative solution. So we may write

$$(2) \qquad f = \sum_{j=1}^{n} \gamma_j (F)^j, \qquad \gamma_j \geqq 0.$$

It follows from (2) that there exists j_1, $1 \leqq j_1 \leqq n$, such that

$$(3) \qquad \gamma_{j_1} \neq 0, \quad ((F)^{j_1})_{j_0} \neq 0, \quad \frac{((F)^{j_1})_q}{((F)^{j_1})_{j_0}} \leqq \frac{(f)_q}{(f)_{j_0}}.$$

On the other hand, from (1) we have $(f)_q < ((F)^{j_0})_q$ and $(f)_{j_0} = ((F)^{j_0})_{j_0}$, and so

$$(4) \qquad \frac{(f)_q}{(f)_{j_0}} < \frac{((F)^{j_0})_q}{((F)^{j_0})_{j_0}}.$$

Again, by (1),

$$(5) \qquad (f)_i = 0, \qquad 1 \leqq i \leqq j_0 - 1, \quad q+1 \leqq i \leqq r.$$

Thus by (2) and (5), and the fact that $\gamma_{j_1} \neq 0$, we obtain

$$(6) \qquad ((F)^{j_1})_i = 0, \qquad 1 \leqq i \leqq j_0 - 1, \quad q+1 \leqq i \leqq r.$$

Therefore, if we replace $(F)^{j_0}$ by $(F)^{j_1}$, we shall get a smaller value of α by (3), (4) and (6) except when $((F)^{j_1})_q = 0$ and $F_{q,j} = 0$, $1 \leqq j \leqq q-1$, $j \neq j_0$. In the latter case we get a smaller value of q. Thus in each case we arrive at a contradiction. This completes the proof.

SUBLEMMA 3.3. *Let A be an $m \times n$ matrix of rank r. Let S be a positive definite symmetric bilinear form over $\mathbb{R}^m$. Then there exists a subset Λ of cardinality r of $\{1, 2, \cdots, m\}$ such that*

$$\langle e_i | i \in \Lambda \rangle \cap R(A)^{\perp s} = 0$$

and

$$\forall i \in \Lambda, \quad \exists j, \quad 1 \leqq j \leqq n, \quad A_{i,j} \neq 0.$$

Proof. Let

$$\Delta = \{i | 1 \leqq i \leqq m, \exists j, 1 \leqq j \leqq n, \text{such that } A_{i,j} \neq 0\}.$$

Then

$$R(A) \subseteq \langle e_i | i \in \Delta \rangle$$

and so

$$\dim (R(A)^{\perp s} \cap \langle e_i | i \in \Delta \rangle) = (\text{card } \Delta) - r.$$

By choosing Λ to be a maximal subset of Δ such that $\langle e_i | i \in \Lambda \rangle \cap R(A)^{\perp s} = 0$, we get the desired conclusion.

We now state without proof some basic facts contained in the following two sublemmas.

SUBLEMMA 3.4. *Let A be an $m\times n$ matrix, and let b be a vector of size m. Let S be a positive definite symmetric bilinear form over $\mathbb{R}^m$. Let $b=b_1+b_2$, $b_1\in R(A)$, $b_2\in R(A)^{\perp s}$. Then x_0 is a best approximate solution of the system $Ax=b$ with respect to S if and only if $Ax_0=b_1$.*

SUBLEMMA 3.5. *Let A be an $m\times n$ matrix, and b be a vector of size m. Let S be a positive definite symmetric bilinear form over $\mathbb{R}^m$. Let P, Q be permutation matrices of orders m, n respectively. Then x_0 is a best approximate solution of the system $Ax=b$ with respect to S if and only if $Q^{-1}x_0$ is a best approximate solution of the system $(PAQ)x=Pb$ with respect to PSP^{-1}.*

LEMMA 3.6. *Let A be an $m\times n$ nonnegative matrix of rank r. Let S be a positive definite symmetric bilinear form over $\mathbb{R}^m$. Suppose that $Ax=b$ has a nonnegative best approximate solution with respect to S for every $b\geqq 0$.*

Then there exist permutation matrices P, Q such that

$$(PAQ)_{i,i}\neq 0,\qquad 1\leqq i\leqq r,$$

$$(PAQ)_{i,j}=0,\qquad 1\leqq i\leqq r,\quad 1\leqq j\leqq r,\quad i\neq j,$$

$$\langle e_i|1\leqq i\leqq r\rangle\cap R(PAQ)^{\perp PSP^{-1}}=0.$$

Proof. Let Λ be as in the conclusion of Sublemma 3.3. Without any loss of generality, we may assume $\Lambda=\{1,2,\cdots,r\}$. Clearly, for each $k\notin\Lambda$, $1\leqq k\leqq m$, there exists a unique vector q_k of $R(A)^{\perp s}$ such that

$$(q_k)_i=\begin{cases}1, & i=k,\\ 0, & i\neq k,\end{cases}\qquad r+1\leqq i\leqq m.$$

In order to prove our lemma, it suffices to prove that A satisfies the hypothesis of Lemma 3.2. Let $Z_1(u)$, $Z(u)$ be as in Lemma 3.2. By way of contradiction, let $(A)^{l_1}$ and $a=\sum_{l\in L}(A)^l$ be such that

$$Z_1((A)^{l_1})\subseteq Z_1(a)\quad\text{but}\quad Z((A)^{l_1})\not\subseteq Z(a).$$

We choose $z\in R(A)$ such that $q_k+z\geqq 0$ for all $k\in\{r+1,\cdots,m\}$. Further, for each vector $u\in\mathbb{R}^m$, let

$$T(u)=\{k|r+1\leqq k\leqq m,\ k\notin Z(a),\ (u)_k<0\}.$$

Assume $T(u)\neq\varnothing$. Let us set

$$p(u)=\min T(u).$$

Next we choose positive number $\alpha(u)$ such that

$$(u+\alpha(u)(q_{p(u)}+z))_{p(u)}=0.$$

Now let $u_0=-(A)^{l_1}$, and define v_i, w_i, u_i inductively by

$$v_i=\alpha(u_{i-1})q_{p(u_{i-1})},\quad w_i=\alpha(u_{i-1})z,\quad u_i=u_{i-1}+v_i+w_i.$$

We continue until $T(u_t) = \varnothing$ for some positive integer t. For each $i = 1, 2, \cdots, t$, we have

$$T(u_i) \subset T(u_{i-1}),$$

$$(u_i)_{p(u_{i-1})} = 0,$$

$$(u_{i-1} + w_i)_{p(u_{i-1})} < 0,$$

$$(u_i)_k \geqq 0, \qquad 1 \leqq k \leqq r, \quad k \notin Z_1(a).$$

Let $v = \sum_{i=1}^{t} v_i$, $w = u_0 + \sum_{i=1}^{t} w_i$, and $u = u_t$. Let us also write $p = p(u_{t-1})$ for convenience. Then

$$u = v + w, \qquad v \in R(A)^{\perp_S}, \quad w \in R(A),$$

$$(u)_k \geqq 0 \quad \text{for all } k \notin Z(a), \quad 1 \leqq k \leqq m,$$

$$(w)_p < 0,$$

By the definition of $Z(a)$, there exists $\beta > 0$ such that $\beta a + u \geqq 0$. Further, since $p \notin Z(a)$, $(\beta a + w)_p = (w)_p < 0$. This implies that $Ax = \beta a + u$ does not have any nonnegative best approximate solution with respect to the norm S, a contradiction. Hence A satisfies the hypothesis of Lemma 3.2, completing the proof.

THEOREM 3.7. *Let A be an $m \times n$ matrix of rank r. Let S be a positive definite symmetric bilinear form over $\mathbb{R}^m$ satisfying*

$$(**) \qquad S(e_i, e_k) \geqq 0, \qquad 1 \leqq i \leqq m, \quad 1 \leqq k \leqq m.$$

Suppose $Ax = b$ has a nonnegative best approximate solution with respect to S for every $b \geqq 0$. Then there exist permutation matrices P, Q such that

$$PAQ = \begin{bmatrix} J & JD \\ 0 & 0 \end{bmatrix},$$

where

$$J = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_r \end{bmatrix},$$

z_i is a positive vector of size λ_i and D is some nonnegative matrix; or equivalently A has a nonnegative $\{1, 3\}$-inverse. (The zero block row in the description of PAQ may be absent.)

Proof. By Lemma 3.6 there exist permutation matrices P, Q such that

$$(PAQ)_{i,i} \neq 0, \qquad 1 \leqq i \leqq r,$$

$$(PAQ)_{i,j} = 0, \qquad 1 \leqq i, j \leqq r, \quad i \neq j$$

and that

$$\langle e_i | 1 \leqq i \leqq r \rangle \cap R(PAQ)^{\perp_{PSP^{-1}}} = 0.$$

For each $r + 1 \leqq k \leqq m$, we define q_k to be the unique vector in $R(PAQ)^{\perp_{PSP^{-1}}}$ such that

$$(q_k)_k = 1, \qquad (q_k)_i = 0, \quad i \neq k, \quad r + 1 \leqq i \leqq m.$$

For each $k \in \{r + 1, \cdots, m\}$, let

$$p_k = \text{card}\,\{j | 1 \leqq j \leqq r, (PAQ)_{k,j} \neq 0\}.$$

We have only to show that $p_k \leqq 1$, $r+1 \leqq k \leqq m$. By way of contradiction, suppose $p_{k_0} \geqq 2$ for some k_0. By (**), there exists i_0, $1 \leqq i_0 \leqq r$, such that

$$(q_{k_0})_{i_0} < 0.$$

Clearly, there exist nonnegative numbers α_j's, $r+1 \leqq j \leqq m$, $j \neq k_0$, such that

$$\Big(-(PAQ)^{i_0} + \sum_{\substack{r+1 \leqq j \leqq m \\ j \neq k_0}} \alpha_j q_j\Big)_k \geqq 0$$

for all $k \in \{r+1, \cdots, m\}$, $k \neq k_0$. Since $(q_{k_0})_{i_0} < 0$, there exists $\alpha_{k_0} < 0$ such that

$$\Big(-(PAQ)^{i_0} + \sum_{r+1 \leqq j \leqq m} \alpha_j q_j\Big)_k \geqq 0$$

for all $k \in \{r+1, r+2, \cdots, m\}$ with $k \neq k_0$ and for $k = i_0$. Since $p_{k_0} \geqq 2$, there exists j_0, $1 \leqq j_0 \leqq r$, $j_0 \neq i_0$ such that $(PAQ)_{k_0, j_0} \neq 0$. Then there exists $\beta > 0$ such that

$$\Big(-(PAQ)^{i_0} + \beta(PAQ)^{j_0} + \sum_{r+1 \leqq j \leqq m} \alpha_j q_j\Big)_k \geqq 0$$

for all $k \in \{r+1, r+2, \cdots, m\}$ and for $k = i_0$. Finally, there exist nonnegative numbers γ_j's, $1 \leqq j \leqq r$, $j \neq i_0$, such that

$$b = -(PAQ)^{i_0} + \beta(PAQ)^{j_0} + \sum_{\substack{1 \leqq j \leqq r \\ j \neq i_0}} \gamma_j (PAQ)^j + \sum_{r+1 \leqq j \leqq m} \alpha_j q_j \geqq 0.$$

Since

$$\Big(b - \sum_{r+1 \leqq j \leqq m} \alpha_j q_j\Big)_{i_0} = -(PAQ)_{i_0, i_0} < 0,$$

the equation $PAQx = b$ does not have any nonnegative best approximate solution with respect to the bilinear form PSP^{-1}. Hence, by Sublemma 3.5 the system $Ax = P^{-1}b$ does not have any nonnegative best approximate solution with respect to S, a contradiction. This gives us the desired structure of A. The last statement follows from the theorem of Berman–Plemmons [3, Thm. 5].

We now proceed to give certain remarks about sufficiency conditions in order that for all b, $Ax = b$ have a nonnegative best approximate solution.

Remark 3.8. Let A be a nonnegative $m \times n$ matrix of the form

$$A = \begin{bmatrix} J & JD \\ 0 & 0 \end{bmatrix},$$

where J and D are as in the statement of Theorem 3.7. Let S be a positive definite symmetric bilinear form satisfying $S(e_i, e_k) \geqq 0$, $1 \leqq i$, $k \leqq m$. Then the following two statements are equivalent:

(i) $Ax = b$ has a nonnegative best approximate solution w.r.t. S for all nonnegative vectors $b \in \mathbb{R}^m$.

(ii) For each $v \in R(A)^{-s}$, either there exists $k \geqq (\sum_{i=1}^{r} \lambda_i) + 1$ such that $(v)_k < 0$ or

$$\forall j, \quad 1 \leqq j \leqq r, \quad \exists k_j \quad \text{with} \Big(\sum_{i=1}^{j-1} \lambda_i\Big) + 1 \leqq k_j \leqq \sum_{i=1}^{j} \lambda_i$$

such that $(v)_{k_j} \leqq 0$.

Proof. Straightforward.

Remark 3.9. Condition (ii) in the above remark is automatically satisfied if S is diagonal. Thus, for such an S, the converse of Theorem 3.7 also holds.

4. Nonnegative least squares solution. The characterization of nonnegative idempotent matrices plays an important role in this section. We state this in the following lemma due to Flor [4].

LEMMA 4.1. [4, Thm. 2]. *Let B be a nonnegative idempotent matrix of rank s. Then there exists a permutation matrix P such that*

$$PBP' = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a direct sum of matrices $x_i y_i'$, where x_i, y_i are positive vectors such that $y_i' x_i = 1$, $1 \leqq i \leqq s$ and C, D are nonnegative matrices of suitable sizes.

The lemma that follows characterizes all real matrices A which commute with a nonnegative idempotent matrix B such that rank AB = rank A.

LEMMA 4.2. *Let B be an idempotent matrix of rank s of the form*

$$\begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where diagonal blocks are square matrices of orders a_1, a_2, a_3, a_4, J is a direct sum of $m_i \times m_i$ matrices $x_i y_i'$ with $y_i' x_i = 1$ and x_i, y_i having no zero entry, $1 \leqq i \leqq s$. Let A be a square matrix such that $AB = BA$ and rank AB = rank A. *Then*

$$A = \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & CKD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal blocks are square matrices of orders a_1, a_2, a_3, a_4 and $K = (K_{ij})$, $1 \leqq i, j \leqq s$ where $K_{ij} = \beta_{ij} x_i y_j'$ is an $m_i \times m_j$ block matrix.

Furthermore, $AB = A = BA$.

Proof. Let $x_{i,j}$ and $y_{i,j}$ denote the ith entry of x_j and y_j respectively. Set

$$n_i = \sum_{j=1}^{i-1} m_j, \quad 2 \leqq i \leqq s, \qquad n_1 = 0.$$

Then

$$(B)^{n_i + l_1} = \frac{y_{l_1,i}}{y_{l_2,i}} (B)^{n_i + l_2}, \qquad 1 \leqq i \leqq s, \quad 1 \leqq l_1, l_2 \leqq m_i.$$

Therefore,

$$(AB)^{n_i + l_1} = \frac{y_{l_1,i}}{y_{l_2,i}} (AB)^{n_i + l_2}, \qquad 1 \leqq i \leqq s, \quad 1 \leqq l_1, l_2 \leqq m_i. \tag{7}$$

Now, we have

$$(B)^l = \sum_{k=1}^{a_1} d_{kl}(B)^k, \tag{8}$$

where

$$d_{kl} = \begin{cases} \delta_{kl}, & l \leqq a_1, \\ (k, l-a_1)\text{-entry of } D, & a_1+1 \leqq l \leqq a_1+a_2, \\ 0, & l \geqq a_1+a_2+1. \end{cases}$$

It follows from (8) that

$$(AB)^l = \sum_{k=1}^{a_1} d_{kl}(AB)^k. \tag{9}$$

By (7) and (9), we obtain

$$R(AB) = \langle (AB)^{n_i+1} \mid 1 \leqq i \leqq s \rangle. \tag{10}$$

Let $r = \operatorname{rank}(AB)$. Since $AB = BA$, (10) implies that we can choose r linearly independent vectors among $(BA)^{n_i+1}$, $1 \leqq i \leqq s$. By simultaneous rearrangement of rows and columns, we may assume that $(BA)^{n_i+1}$, $1 \leqq i \leqq r$, are linearly independent. Then since $(BA)^l = B(A)^l$, we get that $(A)^{n_i+1}$, $1 \leqq i \leqq r$, are linearly independent and, hence, form a basis of $R(A)$. Therefore an arbitrary column $(A)^l$ of A can be expressed as

$$(A)^l = \sum_{i=1}^{r} \alpha_{li}(A)^{n_i+1}. \tag{11}$$

Then

$$(BA)^l = \sum_{i=1}^{r} \alpha_{li}(BA)^{n_i+1}. \tag{12}$$

From (7) and (12),

$$\alpha_{n_j+l_1,i} = \frac{y_{l_1,j}}{y_{l_2,j}} \alpha_{n_j+l_2,i}.$$

The above together with (11) yields

$$(A)^{n_j+l_1} = \frac{y_{l_1,j}}{y_{l_2,j}} (A)^{n_j+l_2}. \tag{13}$$

Similarly,

$$(A)_{n_j+l_1} = \frac{x_{l_1,j}}{x_{l_2,j}} (A)_{n_j+l_2}. \tag{14}$$

Set

$$\beta_{ij} = \frac{(n_i+1, n_j+1)\text{-entry of } A}{x_{1,i} y_{1,j}}.$$

This gives the desired structure for K. Now let $l \geqq a_1 + 1$. Then

$$
\begin{aligned}
(BA)^l &= \sum_{k=1}^{a_1} d_{kl}(BA)^k \qquad \text{(from (9))} \\
&= \sum_{k=1}^{a_1} d_{kl} \sum_{i=1}^{r} \alpha_{ki}(BA)^{n_i+1} \qquad \text{(from (12))} \\
&= \sum_{i=1}^{r} \left(\sum_{k=1}^{a_1} d_{kl}\alpha_{ki} \right)(BA)^{n_i+1}.
\end{aligned}
$$

Thus from (12)

$$\alpha_{li} = \sum_{k=1}^{a_1} d_{kl}\alpha_{ki}.$$

Hence by (11)

$$
\begin{aligned}
(A)^l &= \sum_{i=1}^{r} \left(\sum_{k=1}^{a_1} d_{kl}\alpha_{ki} \right)(A)^{n_i+1} = \sum_{k=1}^{a_1} d_{kl}\left(\sum_{i=1}^{r} \alpha_{ki}(A)^{n_i+1} \right) \\
&= \sum_{k=1}^{a_1} d_{kl}(A)^k \quad \text{(from (11))}.
\end{aligned}
$$

Let X be the submatrix of A consisting of its first a_1 columns. Then by the above equation, and by the definition of d_{kl}, we obtain

$$A = [X \quad XD \quad 0 \quad 0].$$

A similar argument for rows yields

$$X = \begin{bmatrix} K \\ 0 \\ CK \\ 0 \end{bmatrix}.$$

Hence A is of the desired form. The last statement is obvious. This completes the proof.

The $s \times s$ matrix (β_{ij}) in the above lemma will be referred to as a coefficient matrix of A with respect to B. More generally, let B be an arbitrary nonnegative idempotent matrix, and let P be a permutation matrix such that PBP' is as in Lemma 4.1. Let A be a nonnegative matrix such that $AB = BA$ and rank $(AB) =$ rank A. Then we can define a coefficient matrix (β_{ij}) of PAP' with respect to PBP'. We refer to this matrix (β_{ij}) also as a coefficient matrix of A with respect to B. We remark that this definition of coefficient matrix of A is unique up to similarity by a monomial matrix. For, if A, B and P are as above and if we write

$$PBP' = U'_P V''_P,$$

where

$$U'_P = \begin{bmatrix} U \\ 0 \\ CU \\ 0 \end{bmatrix}, \qquad U_P = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_s \end{bmatrix},$$

$$V'_P = \begin{bmatrix} V \\ D'V \\ 0 \\ 0 \end{bmatrix}, \qquad V_P = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_s \end{bmatrix},$$

then the coefficient matrix defined above is the $s \times s$ matrix A_P such that $PAP' = U'_P A_P V''_P$. Note that U_P and V_P are not unique even if P is fixed, but that if P, U_P, V_P are all fixed, then A_p is uniquely determined. Now suppose that for some permutation matrix Q, $QBQ' = U'_Q V''_Q$ is also as in Lemma 4.1. Then the matrix A_Q such that $QAQ' = U'_Q A_Q V''_Q$ is obtained as follows:

$$\begin{aligned} QAQ' &= QP^{-1}U'_P A_P V''_P (QP^{-1})' \\ &= (QP^{-1}U'_P Q_0^{-1})(Q_0 A_P Q_0^{-1})(Q_0 V''_P (QP^{-1})'), \end{aligned}$$

where Q_0 is the unique $s \times s$ monomial matrix such that $QP^{-1}U'_P Q_0^{-1} = U'_Q$ (or equivalently, $QP^{-1}V'_P Q'_0 = V'_Q$.) Thus,

$$A_Q = Q_0 A_P Q_0^{-1}.$$

This justifies our remark that the coefficient matrix is determined up to similarity by a monomial matrix. Before proving our next main result, we fix the following notation.

Let B be a nonnegative idempotent matrix of rank s. We shall without any loss of generality assume (by using Lemma 4.1) that B is of the form

$$\begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a matrix as in Lemma 4.1. Let $C = (c_{ij})$, $1 \leqq i \leqq a_3$, $1 \leqq j \leqq a_1$. Let $x_{i,j}$, $y_{i,j}$, m_i, n_j be as in proof of Lemma 4.2. Set

$$g_{jk} = \sum_{i=1}^{m_k} (c_{j,n_k+i} x_{i,k}), \qquad 1 \leqq j \leqq a_3, \quad 1 \leqq k \leqq s,$$

$$h_{kl} = \sum_{j=1}^{a_3} g_{jk} g_{jl}, \qquad 1 \leqq k, l \leqq s.$$

Let S be an $s \times s$ symmetric matrix given by

$$S_{k,l} = h_{kl} + \delta_{kl} \|x_k\|_2^2.$$

Then

$$z' S z = \sum_{j=1}^{s} (\|x_j\|_2 z_j)^2 + \sum_{j=1}^{a_3} \left(\sum_{k=1}^{s} g_{jk} z_k \right)^2, \qquad z \in \mathbb{R}^s,$$

and therefore the symmetric bilinear form defined by S is positive definite. We also note that S is diagonal if and only if $C = 0$.

THEOREM 4.3. *Let B be the matrix as above, and let A be a nonnegative matrix such that $AB = BA$ and* rank AB = rank A. *Let $A_0 = (\beta_{ij})$, $1 \leqq i, j \leqq s$, be the coefficient matrix of A with respect to B described in Lemma* 4.2.

Then $Ax = b$ has a nonnegative least squares solution for all nonnegative vectors $b \in R(B)$ if and only if $A_0 x = b$ has a nonnegative best approximate solution with respect to S for all nonnegative vectors $b \in \mathbb{R}^s$, where S is the symmetric bilinear form defined above.

Proof. As stated above, we assume

$$B=\begin{bmatrix} J & JD & 0 & 0\\ 0 & 0 & 0 & 0\\ CJ & CJD & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where C, J, D are as in Lemma 4.1.

Let

$$u=\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_s\end{bmatrix}\in\mathbb{R}^s.$$

Define $u^\mu\in\mathbb{R}^{a_1}$, u^λ, $u^\nu\in\mathbb{R}^{a_1+a_2+a_3+a_4}$ by

$$(u^\lambda)_i=\begin{cases} u_l, & i=n_l+1,\quad 1\leqq l\leqq s,\\ 0 & \text{otherwise,}\end{cases}$$

i.e., $u^\lambda=[u_1 0\cdots 0\, u_2 0\cdots 0\, u_s 0\cdots 0]^t$,

$$u^\mu=\begin{bmatrix} u_1x_1\\ u_2x_2\\ \vdots\\ u_sx_s\end{bmatrix},\qquad u^\nu=\begin{bmatrix} u^\mu\\ 0_{a_2}\\ Cu^\mu\\ 0_{a_4}\end{bmatrix},$$

where x_i, $1\leqq i\leqq s$, are the vectors appearing in the representation of the matrix B. We note that ν is an isomorphism from $\mathbb{R}^s$ onto $R(B)$, and indeed, ν maps nonnegative vectors in $\mathbb{R}^s$ to nonnegative vectors in $R(B)$.

Let

$$v=\begin{bmatrix} v_1\\ v_2\\ \vdots\end{bmatrix}\in\mathbb{R}^{a_1+a_2+a_3+a_4}.$$

We define $v^\phi\in\mathbb{R}^s$, $v^\psi\in\mathbb{R}^{a_1+a_2+a_3+a_4}$ as follows:

$$(v^\phi)_l=v_{n_l+1},\quad 1\leqq l\leqq s,\qquad v^\psi=((X^{-1}SX^{-1})v^\phi)^\lambda,$$

where X is an $s\times s$ matrix such that $X_{k,l}=\delta_{kl}x_{1,k}$. We claim

$$(u^\nu)^\phi=Xu,\qquad u\in\mathbb{R}^s \tag{15}$$

$$((u^\nu)^\psi)^\phi=X^{-1}Su,\qquad u\in\mathbb{R}^s, \tag{16}$$

$$v-v^\psi\in R(B)^\perp,\qquad v\in R(B). \tag{17}$$

Since $(u^\nu)_{n_l+1}=x_{1,l}u_l$, $1\leqq l\leqq s$, claim (15) follows immediately. Further, since $(v^\psi)^\phi=(X^{-1}SX^{-1})v^\phi$, claim (16) follows from claim (15). We now proceed to prove claim (17). Since $\{(e_l)^\nu|1\leqq l\leqq s\}$ is clearly a basis of $R(B)$, and since the operation ψ is linear, it suffices to prove the claim (17) for $v=(e_l)^\nu$. By definition of ν, we have

$$((e_l)^\nu)_i=\begin{cases} x_{j,l}, & i=n_l+j,\quad 1\leqq j\leqq m_l,\\ g_{jl}, & i=a_1+a_2+j,\quad 1\leqq j\leqq a_3,\\ 0 & \text{otherwise.}\end{cases}$$

By (16),

$$(((e_l)^\nu)^\psi)_i = \begin{cases} \frac{1}{x_{1,k}}(h_{kl} + \delta_{kl}\|x_k\|_2^2), & i = n_k + 1, \quad 1 \leqq k \leqq s \\ 0 & \text{otherwise.} \end{cases}$$

By actual computations we obtain

$$(((e_l)^\nu)^\psi)'(e_k)^\nu = h_{kl} + \delta_{kl}\|x_k\|_2^2 = \sum_{j=1}^{a_3} g_{jl}g_{jk} + \delta_{kl}\sum_{i=1}^{m_k}(x_{i,k})^2 = ((e_l)^\nu)'(e_k)^\nu.$$

Therefore

$$((e_l)^\nu - ((e_l)^\nu)^\psi)'(e_k)^\nu = 0, \qquad 1 \leqq l, k \leqq s.$$

Hence

$$(e_l)^\nu - ((e_l)^\nu)^\psi \in R(B)^\perp, \qquad 1 \leqq l \leqq s.$$

This proves our claim (17).

Now assume $Ax = b$ has a nonnegative least squares solution for all nonnegative vectors $b \in R(B)$, and let c be an arbitrary nonnegative vector in $\mathbb{R}^s$. Since $c^\nu \in R(B)$, $Ax = c^\nu$ has a nonnegative least squares solution, say

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}.$$

Then

$$c^\nu = \sum_{i=1}^{a_1+a_2+a_3+a_4} f_i(A)^i + w_1, \qquad w_1 \in R(A)^\perp. \tag{18}$$

Further, since each $(A)^i$ can be expressed as a nonnegative linear combination of $(A)^{n_1+1}, \cdots, (A)^{n_s+1}$, we may assume in (18) that $f_i = 0$, $i \neq n_l + 1$, $1 \leqq l \leqq s$.

Next, we claim

$$(A)^{n_l+1} = y_{1,l}((A_0)^l)^\nu, \qquad 1 \leqq l \leqq s. \tag{19}$$

To prove (19) let

$$d_l = \begin{bmatrix} \beta_{1l}x_1 \\ \beta_{2l}x_2 \\ \vdots \\ \beta_{sl}x_s \end{bmatrix}.$$

Then

$$(A)^{n_l+1} = \begin{bmatrix} y_{1,l}d_l \\ 0 \\ Cy_{1,l}d_l \\ 0 \end{bmatrix}.$$

Therefore,

$$(A)^{n_l+1} = y_{1,l}((A_0)^l)^\nu, \qquad 1 \leqq l \leqq s,$$

as desired.

Then, by (18), (19) and by the assertion following (18), we have

(20) $$c' = \sum_{l=1}^{s} f_{n_l-1} y_{1,l}((A_0)^l)' + w_1.$$

Set

$$w_2 = c' - (c')^{\psi},$$
$$z_l = ((A_0)^l)^{\nu} - (((A_0)^l)^{\nu})^{\psi}, \qquad 1 \leqq l \leqq s.$$

Then by (17) $w_2, z_l \in R(B)^{-} \subseteq R(A)^{-}$. Also, by (20),

(21) $$(c')^{\psi} = \sum_{l=1}^{s} f_{n_l-1} y_{1,l}(((A_0)^l)^{\nu})^{\psi} + w_3,$$

where

$$w_3 = w_1 + \left(\sum_{l=1}^{s} f_{n_l-1} y_{1,l} z_l \right) - w_2,$$

and thus $w_3 \in R(A)^{-}$. Set $w' = (X^{-1}S)^{-1}(w_3)^{\phi}$. By (16) and (21),

$$(X^{-1}S)c = \sum_{l=1}^{s} f_{n_l-1} y_{1,l}((X^{-1}S)(A_0)^l) + (X^{-1}S)w'.$$

It then follows that

(22) $$c = \sum_{l=1}^{s} f_{n_l-1} y_{1,l}(A_0)^l + w'.$$

Since $w_3 \in R(A)^{-}$, we get from (19),

(23) $$(((A_0)^l)^{\nu})' w_3 = 0, \qquad 1 \leqq l \leqq s.$$

Also, by (21), $(w_3)_i = 0$, $i \neq n_l + 1$, $1 \leqq l \leqq s$. Therefore, we may rewrite (23) as

(23') $$(X(A_0)^l)'(w_3)^{\phi} = 0$$

by using (15). Then by (23'), together with the definition of w', we have

$$((A_0)^l)' S w' = 0.$$

Hence, $w' \in R(A_0)^{\perp_S}$. Thus (22) gives us a nonnegative best approximate solution of $A_0 x = c$ with respect to the norm S.

We can retrace the steps back to prove the "if" part of the theorem, completing the proof.

Combining Theorems 3.7 and 4.3, we obtain the following main result.

THEOREM 4.4. *Let B be a nonnegative idempotent matrix. Let A be a nonnegative matrix such that $AB = BA$ and* rank $(AB) =$ rank A. *Let A_0 be a coefficient matrix of A with respect to B. Suppose that the equation $Ax = b$ has a nonnegative least squares solution for all nonnegative vectors $b \in R(B)$. Then there exist permutation matrices P, Q such that A_0 can be expressed in the form*

$$PA_0Q = \begin{bmatrix} G & GL \\ 0 & 0 \end{bmatrix},$$

where

$$G = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_k \end{bmatrix},$$

z_i are positive vectors and L is a nonnegative matrix, or equivalently, A_0 has a nonnegative $\{1, 3\}$-inverse.

We give an example to demonstrate that the converse of Theorem 4.4 is not necessarily true.

Example 4.5. Let

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 \end{bmatrix}.$$

Then $B = B^2$, $AB = BA$, rank $(AB) =$ rank $A = 2$ and a coefficient matrix A_0 of A is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Although A_0 has a nonnegative $\{1, 3\}$-inverse, we may verify that the system $Ax = b$, where

$$b = \begin{bmatrix} 0 \\ 13 \\ 0 \\ 13 \end{bmatrix} \in R(B),$$

does not possess a nonnegative least squares solution. For, if we write $b = b_1 + b_2$ where

$$b_1 = \begin{bmatrix} 5 \\ 5 \\ -1 \\ 14 \end{bmatrix} \in R(A), \qquad b_2 = \begin{bmatrix} -5 \\ 8 \\ 1 \\ -1 \end{bmatrix} \in R(A)^{\perp},$$

then by Sublemma 3.4 a least squares solution x_0 must satisfy $Ax_0 = b_1$. But then x_0 cannot be nonnegative.

Remarks 4.6. (1) Recall from Remark 3.9 that if the positive definite symmetric bilinear form S is diagonal, then the existence of a nonnegative $\{1, 3\}$-inverse of a matrix A is equivalent to the existence of nonnegative best approximate solution of $Ax = b$ for all nonnegative vectors b. Also recall that the symmetric bilinear form S in Theorem 4.3 is diagonal if and only if the matrix C in Lemma 4.1 is zero. Therefore, it follows that the converse of Theorem 4.4 holds if $C = 0$.

(2) Example 4.5 shows that the converse of Theorem 4.4 does not hold. Nevertheless, we can show that if an $m \times m$ matrix A is as in the conclusion of Theorem 4.4, then there exists a positive definite symmetric bilinear form S over $\mathbb{R}^m$ such that

$Ax = b$ has a nonnegative best approximate solution with respect to S for all nonnegative vectors $b \in R(B)$.

(3) Let B be an $m \times n$ (not necessarily square) nonnegative matrix of rank s such that

$$(P_0BQ_0)_{i,i} \neq 0, \quad 1 \leqq i \leqq s, \qquad (P_0BQ_0)_{i,j} = 0, \quad 1 \leqq i, j \leqq s, \quad i \neq j$$

for suitable permutation matrices P_0 and Q_0. Let A be an $m \times l$ nonnegative matrix such that $R(A) \subseteq R(B)$. Let A_0' be the matrix consisting of the first s rows of P_0AQ_0. With A, B and A_0' as above, arguments similar to the proof of Theorem 4.3 prove the following:

If $Ax = b$ has a nonnegative least squares solution for all nonnegative $b \in R(B)$, then A_0' has a nonnegative $\{1, 3\}$-inverse.

In case A and B are as in Theorem 4.4, we give below the relation between a coefficient matrix $A_0 = (\beta_{ij})$ and the matrix A_0'. It follows as a consequence that the existence of a nonnegative $\{1, 3\}$-inverse of A_0' implies that of A_0 and vice versa. Let P be as in Lemma 4.1. Also let the notation be as in the proof of Lemma 4.2 with B replaced by PBP'. Further, let P_1 be a permutation matrix which sends the $(n_i + 1)$th row to the ith row. Set $P_0 = P_1P$ and $Q_0 = P_0'$. Then P_0BQ_0 is in the form stated at the beginning of this remark. With this choice of P_0 and Q_0, we have

$$\beta_{ij} = \frac{(A_0')_{i,j}}{x_{1,i}y_{1,j}}, \qquad 1 \leqq i, j \leqq s.$$

Since every column of A_0' can be written as a nonnegative linear combination of the first s columns of A_0', the existence of a nonnegative $\{1, 3\}$-inverse of one of A_0 or A_0' implies that of the other.

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