THE MATHEMATICS STUDENT
Vol. 46 No. 1 (1978) 42—48

NONNEGATIVE MATRICES HAVING NONNEGATIVE GENERLIZED INVERSES[1]

S.K. Jain

With an excellent article of R. Penrose in 1956 regarding least-squares lutions of a system of linear equations through generalized inverses, there s been a lot of interest in the study of λ-inverses of matrices. Nonnegative atrices having nonnegative λ-inverses are of importance in numerical alysis and linear programming. The purpose of this address is to talk out decomposition theorems for a certain class of nonnegative matrices. e reason for a study of such decomposition theorems is the fact that the composition of matrices into a direct sum of suitable "fine" blocks is torically an important study both for the development of the subject and applications in other branches. Various authors including A. Berman, E. Cline, R. DeMarr, P. Flor, F. Harary, E. Haynsworth, M. Lewin, Minc, R. J. Plemmons, J. R. Wall have studied this question under tain conditions.

For any real or complex matrix A and for any nonempty subset λ of 2,3,4,5}, X is called a λ-inverse (or generalized inverse) of A if X satisfies uation (i) for each $i \in \lambda$ where (1) $AXA=A$, (2) $XAX=X$, (3) $(AX)^*=AX$, $(XA)^*=XA$, (5) $AX=XA$. A λ-inverse of A is denoted by $A^{(\lambda)}$. For {1,2,3,4}, $A^{(\lambda)}$ is unique and is commonly denoted by $A^{\dagger}$ — called Moore-Penrose generalized inverse of A. For $\lambda=\{1,2,5\}$, $A^{(\lambda)}$, if it sts, is also unique and is denoted by $A^{\#}$ — called the group inverse Peter Flor [12] gave a complete characterization of nonnegative idempo- t matrices in the following

THEOREM 1. *Let I be a nonnegative idempotent matrix of rank k. n there exists a permutation matrix P such that*

$$P^{-1}IP=\begin{bmatrix} J & JB & 0 & 0 \\ 0 & 0 & 0 & 0 \\ AJ & AJB & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

[1]Invited address at the 43rd Conference of the Indian Mathematical Society at Aligarh in December, 1977.

where $J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix}$,

where J_i *are nonnegative idempotent matrices of rank* 1.

Theorem of *P.* Flor includes a proof of Brown's theorem [6] as well as an extension of some results of Schwarz [27]. Ralph DeMarr [9] has given another description of nonnegative idempotent matrices based on ideas in the theory of partially ordered linear algebras. *A* kind of "spectral decomposition" is obtained in

THEOREM 2. *Let A be a partially ordered linear algebra of matrices having the Archimedean property. If* $u \epsilon A$ *and* $0 \leqslant u = u^2$ *then* $u = z_1 + z_2 + \ldots + z_m$, *where* $z_k \geqslant 0$, $z_k z_n = \delta_{kn} z_k$, *and for each k the matrix* z_k *is of rank 1.*

Nonnegative matrices having nonnegative λ-inverses (also called λ-monotone matrices) have been considered by many authors including A. Berman, R. Cline, R.J. Plemmons, H. Minc, and F. Harary. F. Harary and H. Minc were the first to look at the decomposition of such matrices and they considered a special case : A nonnegative nonsingular matrix *A* with nonnegative inverse equal to itself. A. Berman [3] extended their result for any nonnegative matrix (not necessarily nonsingular).

THEOREM 3. *Let A be a nonnegative matrix. Then* $A^{\dagger} = A$ *iff A is square and there exists a permutation matrix P such that* PAP^T *is a direct sum of square matrices of the following (not necessarily all) three types :*

(i) xx^T, *where x is a positive unit vector.*

(ii) $\begin{pmatrix} 0 & axy^T \\ byx^T & 0 \end{pmatrix}$,

x, y are positive vectors and $ab = 1$.

(iii) *a zero matrix.*

The theorem of Berman raised many questions : what do you know if $A^{(\lambda)}$ is nonnegative and is equal to some "well-behaved" function of *A* ?

Question 1. *Let* $A \geqslant 0$ *and* $A^{\dagger} = p(A) \geqslant 0$, *where p(A) is some polynomial in A. Characterize A.*

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Question 2. *Let* $A \geqslant 0$ *and* $A^{\#} = p(A) \geqslant 0$, *where* $p(A)$ *is some polynomial in* A. *characterize* A.

More generally,

Question 3. *Let* $A \geqslant 0$ *and* $A^{(\lambda)} = p(A) \geqslant 0$, *where* λ *is any subset of* $\{1,2,3,4,5\}$ *with* $1 \varepsilon \lambda$ *and* $p(A)$ *is a polynomial in* A. *Characterize* A.

A related open question is the following :

Question 4. *Characterize nonnegative matrices forming a finite cyclic semigroup. Equivalently, characterize the class of nonnegative matrices* A *with* $A^{(2)} = A^m$, *where* m *is some positive integer.*

In joint work with V.K. Goel and Edward K. Kwak, a complete answer to the question 1 is given in [16], [17], [18]. We also obtain, in particular, the theorems of F. Harary and H. Minc [14], A. Berman [3], M. Lewin [21].

Regarding question 2, the following theorem in [19] completely answers the question 2.

THEOREM 4. *Let* $A \geqslant 0$ *and* $A^{\#} = p(A) \geqslant 0$, *where*

$$p(A) = \alpha_1 A^{m_1} + \dots + \alpha_k A^{m_k}$$

is some polynomial in A. *Then there is a permutation matrix* P *such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D *are some nonnegative matrices of appropriate sizes and* J *is a direct sum of matrices of the following types (not necessarily both) :*

(i) *matrices of rank* 1 *of the form* $\beta x y^T$, *where* x, y *are positive* vectors *with* $y^T x = 1$, *and* β *is a positive number satisfying*

$$\sum \alpha_i \beta^{m_i+1} = 1.$$

(ii) *matrices of rank* d, $d \neq 1$, $d \mid m_i + 1$ *for some* m_i, *of the form*

$$\begin{pmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 \ \dots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 \ \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \ \dots & \cdot \\ 0 & 0 & 0 \ \dots & 0 \ \beta_{d-1,d}x_{d-1} & y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 \ \dots & 0 & 0 \end{pmatrix},$$

where x_i and y_i are positive vectors of the tame order with $y_i^T\, x_i = 1$; x_i and x_j, $i \neq j$, are not necessarily of the same order, and $\beta_{12}, \beta_{23}, \dots, \beta_{d1}$ are arbitrary positive numbers such that their product is a common root of the following system of at most dequations in t

$$\sum_{d|(m_i+1)} a_i t^{(m_i+1)/d} = 1$$

$$\sum_{d|(m_i+1-k)} a_i t^{(m_i+1-k)/d} = 0, \ k \in \{1, \dots, d-1\}$$

where the summation in each of the above equations runs over all those m_i for which $d|(m_i+1-k)$, $k=0, 1, \dots, d-1$, with the convention that if there is no m_i for which $d|(m_i+1-k)$, $k=1,\dots, d-1$, then the corresponding equation is absent. (Naturally, the possible values of d are divisors of m_i+1. Among these divisors we shall discard those d for which the above system of equations has no common positive solution.)

As a consequence of the above we obtain the following striking results for range-Hermitian matrices and for stochastic matrices.

THEOREM 5. *If A is range-Hermitian nonnegative matrix having nonnegative group inverse, then, $A^\dagger = A^\# = DA^m = A^mD$, where D is a diagonal matrix with all entries on the diagonal as positive and m is a positive integer.*

THEOREM 6. *If A is a nonnegavive stochastic matrix having nonnegative group inverse $A^\#$, then $A^\# = A^m$, where m is a positive integer and $A^\#$ is stochastic.*

Regarding question 3 we quote the following result of Berman and Plemmons (Theorem 4 [5]).

THEOREM. *Let A be a nonnegative matrix with a nonnegative $\{1\}$-inverse X. Then X is of the form $D_1AD_2^T$ where D_1 and D_2 are nonnegative diagonal matrices*

As a consquence it follows that if λ is any subset of $\{1,2,3,4,5\}$ with $1 \varepsilon \lambda$ then $A^{(\lambda)}=p(A)\geqslant 0$ if and only if $A^{(\lambda)}=A^{\dagger}$ or $A^{(\lambda)}=A^{\#}$. Hence questions 1 and 2 together provide answer to question 3.

Question 4 remains open. The special case where the semigroup is a group is studied by Lewin [21]. Lewin's theorem is also a consequence of our theorem (Theorem 4 [17]).

The concept of 0-symmetric matrix which led to the solution of question 2 and consequently to that of that of question 3 is introduced by us in [17]. We call a matrix $A=(a_{ij})$ to be 0-symmetric if $a_{ij}=0$ implies $a_{ji}=0$. In view of this let us consider the following equations :

(1) $AXA = A$,

(2) $XAX = X$,

(3′) AX is 0-symmetric

(4′) XA is 0-symmetric

(5) $AX = XA$

Question 5. *Find all solutions X satisfying* (1), (2), (3′), (4′). *(of* course, $X=A^{\dagger}$ *is a solution*)

Nonnegative matrices having a nonnegative matrix X satisfying (1), (2), (3′), (4′), and (5) are characterized in [18].

All we have said above relates to finite matrices. As (communicated by a numerical analyst) some of the interesting questions arise from infinite nonnegative matrices which are row (or colum) finite. Thus one may inquire which of the above results can be extended to infinite nonnegative matrices A for which $A^{\dagger}$ is meaningful ? (For instance, A could be a nonnegative bounded linear operator in a real Hilbert space with closed range). This inquiry needs to be looked by first considering basic results analogous to the known results in the finite demensional case. Since all our results stated above make use of the characterization of nonnegative symmetric idempotents one must first ask

Question 6. Characterize all nonnegative infiniee 0-symmetric idempotent matrices.

Further, is there an analogoue of Theorem 6 for a certain class of nonnegative infinite matrices ? These questions are open.

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Ohio University
Athens, Ohio 45701
U S.A.

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