SIAM J. ALG. DISC. METH.
Vol. 2, No. 1, March 1981

NONNEGATIVE λ-MONOTONE MATRICES*

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Abstract. In this paper, the structure of nonnegative $m \times n$ matrices A satisfying $AXA = A$, for some nonnegative $n \times m$ matrix X, is obtained. Several equivalent characterizations of such matrices A have been given earlier by Plemmons [Proc. Amer. Math. Soc., 39 (1973), pp. 26–32] and Berman–Plemmons [Linear and Multlinear Algebra, 2 (1974), pp. 161–172]. The structure of matrices given in this paper unifies all the previous known results on λ-monotone matrices where $1 \in \lambda$. The importance of λ-monotonicity to problems in mathematical economics, in probability and statistics, and in numerical linear algebra, is documented in a recent book by Berman and Plemmons [*Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979].

1. Introduction. Let A be an $m \times n$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, and (5) $AX = XA$, where X is an $n \times m$ real matrix and T denotes the transpose. Let λ be a nonempty subset of $\{1, 2, 3, 4, 5\}$. X is called a λ-inverse of A if X satisfies equation (i) for each $i \in \lambda$. A λ-inverse of a matrix A is generally denoted by $A^{(\lambda)}$. A $\{1, 2, 3, 4\}$–inverse of A is the unique *Moore–Penrose inverse* of A and is denoted by A^{+}. A $\{1, 2, 5\}$–inverse of A exists if and only if $m = n$ and rank A = rank A^2, i.e., index $A = 1$, and is denoted by $A^{\#}$.

A matrix $A = (a_{ij})$ is *nonnegative* if $a_{ij} \geqq 0$ for all i, j, and we denote it by $A \geqq 0$. If $a_{ij} > 0$ for all i, j, we write $A > 0$. A nonnegative matrix is called *λ-monotone* if its λ-inverse is nonnegative. If a matrix A is a direct sum of matrices S_i, then S_i's will be called *summands* of A. S_r will denote the *symmetric group on r symbols*, say $\{1, 2, \cdots, r\}$. Diag A shall denote the main diagonal of the matrix A.

Nonnegative matrices have played a significant role in numerical analysis, economics, and Markov chains. The interested reader is referred to a wealth of selected applications of nonnegative matrices to numerical analysis, probability, economics, and operations research in a recent book by Berman and Plemmons [2]. In many of the applications, one is interested in finding nonnegative solutions of the system $Ax = b$, where $A \geqq 0$ and $b \geqq 0$. If $A^{(\lambda)} \geqq 0$ exists with $1 \in \lambda$ and if the system is consistent, then $x = A^{(\lambda)}b$ provides a nonnegative solution for the system. Of course, as is well known, the existence of the nonnegative $\{1\}$–inverse is sufficient but not necessary for obtaining a nonnegative solution to the consistent system. Also, in the case where the system is inconsistent if B is a $\{1, 3\}$–inverse, then $x = Bb$ yields a least squares solution; i.e., the minimum of $\|Ax - b\|_2$ is attained for $x = Bb$.

Theorem 1 gives the structure of matrices $A \geqq 0$ having a nonnegative $\{1\}$–inverse. The representation of matrices A obtained in Theorem 1 provides a new proof for theorems of Plemmons [17], Berman–Plemmons [5], and gives immediately as special cases the theorems of Berman [3], Plemmons–Cline [18], Haynsworth–Wall [10], [11], Jain-Goel-Kwak [12], [13], [14], Lewin [16] and perhaps some others (see Corollaries 3, 4). Our main result makes use of the following:

THEOREM A [7, Theorem 2]. *If E is a nonnegative idempotent matrix of rank r, then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

* Received by the editors March 31, 1980, and in revised form July 18, 1980.
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where J is a direct sum of matrices $x_i y_i^T$, $x_i, y_i > 0$ and $y_i^T x_i = 1$ and C, D are nonnegative matrices of suitable sizes.

THEOREM B [15, Lemma 2]. *Let X, Y be respectively $m \times n$, $n \times m$ nonnegative matrices such that*

$$XY = \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_1 \end{bmatrix}, \qquad YX = \begin{bmatrix} Y_1 & & 0 \\ & \ddots & \\ 0 & & Y_r \end{bmatrix},$$

where X_i, Y_i are positive square matrices of order a_i, α_i respectively.

If $X = (X_{ij})$, $Y = (Y_{ij})$ are partitionings of X, Y respectively such that X_{ii}, Y_{ii} are of orders $a_i \times \alpha_i$, $\alpha_i \times a_i$ respectively, then there exists $\sigma \in S_r$ such that $X_{j\sigma(j)} \neq 0$, $Y_{\sigma(j)j} \neq 0$, $X_{jk} = 0 = Y_{kj}$, for all $k \neq \sigma(j)$.

2. Preliminary results.

LEMMA 1. *Let L, M be nonnegative matrices of orders $m \times n$, $n \times m$, respectively, such that*

$$LM = \begin{bmatrix} K_1 & K_1 D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1 K_1 & C_1 K_1 D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad ML = \begin{bmatrix} K_2 & K_2 D_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_2 K_2 & C_2 K_2 D_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where diagonal blocks are square matrices, C_i, D_i, $i = 1, 2$ are matrices of suitable sizes, diag $K_i > 0$, *and* rank L = rank M = rank LM = rank ML. *Let $L = (L_{ij})$, $M = (M_{ij})$, $1 \leqq i, j \leqq 4$, be partitionings of L, M such that the block multiplication of L with M in either order can be performed. Then*

$$L = \begin{bmatrix} L_{11} & L_{11} Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X L_{11} & X L_{11} Z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{11} Z' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X' M_{11} & X' M_{11} Z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for some matrices Z, X, Z', X' (not necessarily nonnegative) of suitable sizes, and

$$\text{rank } L_{11} = \text{rank } L = \text{rank } M = \text{rank } M_{11} \quad \textit{and} \quad L_{11} M_{11} = K_1, \quad M_{11} L_{11} = K_2.$$

Proof. We have

(1) $$L_{11}M_{11} + L_{12}M_{21} + L_{13}M_{31} + L_{14}M_{41} = K_1,$$

(2) $$M_{11} + L_{11} + M_{12}L_{21} + M_{13}L_{31} + M_{14}L_{41} = K_2,$$

(3) $$L_{1j}M_{j3} = 0, \qquad 1 \leqq j \leqq 4,$$

(3)′ $$M_{1j}L_{j3} = 0, \qquad 1 \leqq j \leqq 4,$$

(4) $$L_{1j}M_{j4} = 0, \qquad 1 \leqq j \leqq 4,$$

(4)′ $$M_{1j}L_{j4} = 0, \qquad 1 \leqq j \leqq 4,$$

(5) $$L_{2j}M_{jk} = 0, \qquad 1 \leqq j, k \leqq 4,$$

(5)′ $$M_{2j}L_{jk} = 0, \qquad 1 \leqq j, k \leqq 4,$$

(6) $$L_{3j}M_{j3} = 0, \qquad 1 \leqq j \leqq 4,$$

(6)′ $$M_{3j}L_{j3} = 0, \qquad 1 \leqq j \leqq 4,$$

(7) $$L_{3j}M_{j4} = 0, \qquad 1 \leqq j \leqq 4,$$

(7)′ $$M_{3j}L_{j4} = 0, \qquad 1 \leqq j \leqq 4,$$

(8) $$L_{4j}M_{jk} = 0, \qquad 1 \leqq j, k \leqq 4,$$

(8)′ $$M_{4j}L_{jk} = 0, \qquad 1 \leqq j, k \leqq 4.$$

Premultiply (1) by M_{21} and use (5)′ to obtain $0 = M_{21}K_1$. Since diag $K_1 > 0$, we get $M_{21} = 0$. Postmultiply (1) by L_{13} and use (3)′, (6)′, (8)′ to get $L_{13} = 0$. Similarly, we get $M_{41} = 0 = L_{14}$.

Similar computations yield

$$L_{21} = 0 = M_{13} = L_{41} = M_{14}.$$

Thus, $L_{11}M_{11} = K_1$, $M_{11}L_{11} = K_2$.

It follows easily that

$$\text{rank } L = \text{rank } L_{11} \equiv \text{rank } K_1.$$

Therefore, for all $i = 2, 3, 4$ there exists a matrix X_i of suitable size such that

(9) $$(L_{i1} \quad L_{i2} \quad L_{i3} \quad L_{i4}) = X_i(L_{11} \quad L_{12} \quad 0 \quad 0).$$

Thus, $L_{i3} = 0 = L_{i4}$, $i = 2, 3, 4$.

Also, rank L = rank L_{11} implies that there exists a matrix Z of suitable size such that

(10) $$\begin{bmatrix} L_{12} \\ L_{22} \\ L_{32} \\ L_{42} \end{bmatrix} = \begin{bmatrix} L_{11} \\ 0 \\ L_{31} \\ 0 \end{bmatrix} Z.$$

But then, $L_{22} = 0 = L_{42}$. Hence,

$$L = \begin{bmatrix} L_{11} & L_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L_{31} & L_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (10) above it follows that $L_{12} = L_{11}Z$, and letting $X_3 = X$ in (9) yields $L_{31} = XL_{11}$ and $L_{32} = XL_{12} = XL_{11}Z$. Similarly,

$$M = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{31} & M_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $M_{12} = M_{11}Z'$, $M_{31} = X'M_{11}$, $M_{32} = X'M_{11}Z'$ for some matrices Z' and X'. This completes the proof.

Lemma 2, which is essentially Theorem B, gives the nature of the submatrices L_{11}, M_{11} of the matrices L, M respectively appearing in Lemma 1.

LEMMA 2. *Let X, Y be $m \times n$, $n \times m$ matrices each of rank r such that*

$$XY = \begin{bmatrix} a_1b_1^T & & 0 \\ & \ddots & \\ 0 & & a_rb_r^T \end{bmatrix}, \qquad YX = \begin{bmatrix} c_1d_1^T & & 0 \\ & \ddots & \\ 0 & & c_rd_r^T \end{bmatrix},$$

where $a_i, b_i, c_i, d_i > 0$, diagonal blocks are square matrices and a_i, a_j (and c_i, c_j), $i \neq j$, are not necessarily of the same sizes. Then:

(1) *There exist permutation matrices P, Q of orders m, n respectively, such that PXQ^T is a direct sum of matrices of types* (I), (II) (*not necessarily both*):

(I) βxy^T, $\beta > 0$, x, y, *are positive unit vectors.*

(II)
$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with all β's > 0; x_i, y_i are positive unit vectors, not necessarily of the same size.

(2) *X has a nonnegative $\{1, 2\}$-inverse.*

Similar results hold for Y.

Furthermore, $P = Q$ if $m = n$.

Proof. The proof, although straightforward, is rather technical and is omitted.

Note. If S is a summand of type (I) then one can verify that

$$\beta^{-1}yx^T$$

is a $\{1, 2\}$-inverse of S, whereas if S is a summand of type (II) then

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_{d1}^{-1}y_1x_d^T \\ \beta_{12}^{-1}y_2x_1^T & 0 & \cdots & 0 & 0 \\ 0 & \beta_{23}^{-1}y_3x_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{d-1,d}^{-1}y_dx_{d-1}^T & 0 \end{bmatrix}$$

is a $\{1, 2\}$-inverse of S.

Henceforth, by matrices of types (I) or (II) we shall mean the matrices of types (I) or (II) described in Lemma 2.

3. Main results.

THEOREM 1. *Let A be a nonnegative $m \times n$ matrix. Then A has a nonnegative $\{1\}$-inverse X if and only if for some permutation matrices P, Q of orders m, n, respectively,*

$$PAQ^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a direct sum of matrices of types (I) *and* (II) (*not necessarily both*), *and C, D are nonnegative matrices of suitable sizes.*

Proof. We have $AXA = A$. This gives $\operatorname{rank} A = \operatorname{rank} AX = \operatorname{rank} XA = \operatorname{rank} XAX = r$, say. Further, since AX and XA are nonnegative idempotents we have, by Flor [7],

$$(11) \qquad P_1AXP_1^T = \begin{bmatrix} K_1 & K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1K_1 & C_1K_1D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

(12)
$$Q_1XAQ_1^T = \begin{bmatrix} K_2 & K_2D_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_2K_2 & C_2K_2D_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where P_1, Q_1 are permutation matrices of orders m, n respectively, C_1, D_1, C_2, D_2 are nonnegative matrices of suitable sizes, and each K_i, $i = 1, 2$, is a square matrix of rank r which is a direct sum of matrices of the form $x_iy_i^T$ and x_i, y_i are positive unit vectors with $y_i^Tx_i = 1$. Set $L = P_1AQ_1^T$, $M = Q_1(XAX)P_1^T$. Then $LM = P_1AXP_1^T$ and $ML = Q_1XAQ^T$. Also, since rank A = rank AX = rank XA = rank XAX, we have

$$\text{rank } L = \text{rank } LM = \text{rank } ML = \text{rank } M.$$

Thus, by Lemma 1, L, M are of the forms

$$L = \begin{bmatrix} L_{11} & L_{11}Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ XL_{11} & XL_{11}Z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{11}Z' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X'M_{11} & X'M_{11}Z' & 0 & 0 \\ 0 & 0 & & 0 \end{bmatrix},$$

where

(13)
$$L_{11}M_{11} = K_1, \; M_{11}L_1 = K_2.$$

Then, by Lemma 2, there exist permutation matrices P_2, Q_2 of suitable orders such that $P_2L_{11}Q_2^T$ is a direct sum of matrices of the types (I), (II) stated in the theorem. Also, by Lemma 2, L_{11} possesses a nonnegative {1}-inverse $L_{11}^{(1)}$. Thus, if we set $L_{11}^{(1)}L_{11}Z = D'$, $XL_{11}L_{11}^{(1)} = C'$, then $D', C' \geqq 0$ and $L_{11}Z = L_{11}D'$, $XL_{11} = C'L_{11}$, $XL_{11}Z = C'L_{11}D'$.

Finally, let us set

(14)
$$P = \begin{bmatrix} P_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} P_1, \qquad Q = \begin{bmatrix} Q_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} Q_1,$$

here the I's are identity matrices of suitable orders such that P, Q are of orders m, n spectively, and the partitionings of P, Q given above are such that the block ultiplication of $PAQ^T = PP_1^TLQ_1Q^T$ can be performed. Then

$$PAQ^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

iere $D = Q_2D'$, $C = C'P_1^T$ are nonnegative matrices of suitable sizes and $J = L_{11}Q_2^T$ is a direct sum of matrices of types (I) and (II) (not necessarily both) as tained above, completing the "only if" part. To prove the "if part" we observe (see e following Lemma 2) that if S denotes a summand of J, then it has a nonnegative

{1}-inverse, $S^{(1)}$. Consequently, J has a nonnegative {1}-inverse $J^{(1)}$. Also, since

$$Q^T\begin{bmatrix} J^{(1)} & J^{(1)}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^{(1)} & CJ^{(1)}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}P$$

is a nonnegative {1}-inverse of A, the converse follows.

COROLLARY 1. *Let* $\lambda = \{1, 5\}$. *Then a nonnegative square matrix* A *is* λ*-monotone if and only if there exists a permutation matrix* P *such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D *are nonnegative matrices of suitable size, diagonal blocks are square matrices and* J *is a direct sum of matrices of the following types (not necessarily both):*

$(\mathrm{I})_*$ βxy^T, $\beta > 0$, x, y *are positive unit vectors of the same size and* $y^Tx = 1$.

$(\mathrm{II})_*$

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with $\beta_{ij} > 0$; x_i, y_i *are positive unit vectors,* x_i, y_i *are of the same size,* x_i, y_j, $i \neq j$ *are not necessarily of the same size and* $y_i^Tx_i = 1$.

Proof. Let X be a nonnegative {1, 5}-inverse of A. Then $AXA = A$, $AX = XA$. Clearly A, X are square matrices of the same order. Thus, in the equations (11) and (12) in the proof of the theorem, we have $P_1 = Q_1$, and L, M (as well as L_{11}, M_{11}) are of the same order. Then by the last statement in Lemma 2, and equation (13) in the theorem, we get $P_2 = Q_2$. Hence, by (14) in the theorem, $P = Q$. Thus,

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of suitable sizes and J is a square matrix. That J is a direct sum of matrices of the types stated in the corollary is obvious, completing the "only if" part of the corollary. The "if part" follows as in the proof of the theorem.

COROLLARY 2. *The class of nonnegative* {1}*-monotone matrices coincides with the class of nonnegative* {1, 2}*-monotone matrices.*

Proof. Let $\mathscr{C}_1$ denote the class of nonnegative {1}-monotone matrices and C_2 denote the class of nonnegative {1, 2}-monotone matrices.

Let $A \in \mathscr{C}_1$. Then, by Theorem 1, there exist permutation matrices P, Q of suitable orders such that

$$PAQ^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $C, D \geqq 0$ and J is a direct sum of matrices of types (I) and (II). Now, if S is a summand of J, then as noted before, S has a nonnegative $\{1, 2\}$-inverse $S^{(1,2)}$. Hence, $J^{(1,2)} \geqq 0$.

Since

$$A^{(1,2)} = P^T \begin{bmatrix} J^{(1,2)} & J^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^{(1,2)} & CJ^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Q,$$

it follows that $A^{(1,2)} \geqq 0$; i.e., $A \in \mathscr{C}_2$. Hence, $\mathscr{C}_1 = \mathscr{C}_2$.

COROLLARY 3. *Let A be a nonnegative matrix and let $A^{(1)} = p(A) \geqq 0$, where $p(A) = \sum_{i=1}^{k} \alpha_i A^{m_i}$, $\alpha_i \neq 0$, $m_i \geqq 0$. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I)$_{**}$ *$\beta x y^T$, where x and y are positive unit vectors with $y^T x = 1$ and β is a positive root of*

$$\sum_{i=1}^{k} \alpha_i t^{m_i+1} = 1. \tag{3}$$

(II)$_{**}$

$$\begin{bmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & \beta_{d-1,d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where x_i, y_i are positive unit vectors of the same order with $y_i^T x_i = 1$; x_i and x_j, $i \neq j$, are not necessarily of the same order. $\beta_{12}, \cdots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d \mid m_i + 1$ for some m_i such that the product $\beta_{12}\beta_{23} \cdots \beta_{d1}$ is a common root of the following system of at most d equations in t:

$$\sum_{d \in \Lambda_0} \alpha_i t^{(m_i+1)/d} = 1,$$

$$\sum_{d \in \Lambda_k} \alpha_i t^{(m_i+1-k)/d} = 0, \qquad k \in \{1, 2, \cdots, d-1\}, \tag{4}$$

where

$$\Lambda_k = \{d : d | m_i + 1 - k, d \neq 1\}, \qquad k = 0, 1, \cdots, d-1,$$

with the understanding that if some $\lambda_k = \emptyset$ *then the corresponding equation is absent.*

Conversely, suppose we have, for some permutation matrix P,

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are arbitrary nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I′) βxy^T, $\beta > 0$, *x, y are positive vectors with* $y^T x = 1$.

(II′)
$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\beta_{ij} > 0$, x_i *and* y_i *are positive vectors with* $y_i^T x_i = 1$. *Then* $A^{(1,2)} \geqq 0$ *and is equal to some polynomial in A with scalar coefficients.*

Proof. By Corollary 1, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a direct sum of matrices of types $(\mathrm{I})_*$ and $(\mathrm{II})_*$ (not necessarily both).

Since $p(A) = \sum_{i=1}^{k} \alpha_i A^{m_i}$ is a {1}-inverse of A,

$$\alpha_1 A^{m_1+2} + \cdots + \alpha_k A^{m_k+2} = A. \tag{15}$$

Also, it is straightforward to verify that if $f(A)$ is any polynomial in A with scalar coefficients, then

$$Pf(A)P^T = \begin{bmatrix} f(J) & f(J)D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Cf(J) & Cf(J)D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, (1) implies

$$\alpha_i J^{m_1+2} + \cdots + \alpha_k J^{m_k+2} = J. \tag{16}$$

Clearly, all summands S of J will also satisfy (2); i.e.,

$$\alpha_1 S^{m_1+2} + \cdots + \alpha_k S^{m_k+2} = S. \tag{17}$$

Then it is a direct verification that if S is a summand of type $(\mathrm{I})_*$ then β must satisfy the equation (a), and if S is a summand of type $(\mathrm{II})_*$ then $\beta_{12}\beta_{23}\cdots\beta_{d1}$ must satisfy the system of equations (b). Hence, J is a direct sum of matrices of the form $(\mathrm{I})_{**}$ and $(\mathrm{II})_{**}$ as desired (for details see [13, Theorem 2]).

Remark 1. The above corollary gives in particular, the earlier known results of Harary-Minc [9], Berman [3], Lewin [16], Jain–Goel–Kwak [12], [13], [14], Haynsworth–Wall [10], [11].

The following theorem giving equivalent characterizations of $\{1\}$-monotone matrices was first proved by Berman–Plemmons and also earlier by Plemmons for square matrices. We show that those characterizations may be obtained as a direct consequence of the structure theorem.

THEOREM 2. [5, Theorem 4]. *For an* $m \times n$ *nonnegative matrix* A *of rank* r *the following are equivalent:*

(a) *A is $\{1\}$-monotone.*

(b) *There exists a $\{1\}$-inverse of the form $D_1A^TD_2$, where D_1, D_2 are nonnegative diagonal matrices.*

(c) *A has a monomial submatrix of rank r.*

(d) *A has a nonnegative rank factorization FG where F, G have monomial submatrices of rank r.*

Proof. (a) $\Rightarrow$ (b). By Theorem 1,

$$PAQ^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for some permutation matrices P, Q, and J is a direct sum of matrices of types (I) and (II) (not necessarily both).

If S is a summand of J, then it can be verified that

$$S^{(1)} = \begin{cases} \beta^{-2}S' & \text{if } S \text{ is of type (I)}, \\ \operatorname{diag}(\beta_{d1}^{-2}, \beta_{12}^{-2}, \cdots, \beta_{d-1,d}^{-2})S^T & \text{if } S \text{ is of type (II)}. \end{cases}$$

Thus, $J^{(1)} = ZJ^T$, where Z is a diagonal matrix. Further, direct verification yields that

$$A^{(1)} = Q^T \begin{bmatrix} J^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P$$

$$= Q^T \begin{bmatrix} ZJ^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P$$

$$= Q^T \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} QQ^T \begin{bmatrix} J^T & 0 & J^TC^T & 0 \\ D^TJ^T & 0 & D^TJ^TC^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P^TP \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P^T$$

$$= D_1A^TD_2,$$

where

$$D_1 = Q^T \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Q,$$

and

$$D_2 = P\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}P^T$$

are nonnegative diagonal matrices. This proves (a)⇒(b). (b)⇒(a) is obvious.

(a)⇔(c). (a)⇒(c) follows at once from the representation of {1}–monotone matrices. (c)⇒(a) is easy and is left to the reader. (Note (c) implies that there are permutation matrices, P, Q such that $PAQ^T = \left[\begin{smallmatrix} M & X \\ Y & Z \end{smallmatrix}\right]$, where rank A = rank M, and $M^{-1} \geqq 0$.)

(a)⇒(d). We appeal to Theorem 1 here again. First we note that if $J = FG$ is a nonnegative rank factorization of J, then it "lifts" to a nonnegative rank factorization

$$A = P^T\begin{bmatrix} F \\ 0 \\ CF \\ 0 \end{bmatrix}(G \quad GD \quad 0 \quad 0)Q$$

of A. Furthermore, it is clear that if S is a summand of J of type (I) or of type (II), then S has a nonnegative rank factorization $S = FG$, such that F and G contain monomials of rank r. This proves (a)⇒(d) (for details see [14, Theorems 3, 4]).

(d)⇒(a). This is well known and is left to the reader.

Remark 2. Using the representation of matrices obtained in Theorem A, the other results of Berman–Plemmons in [5] for λ-monotone matrices where $1 \in \lambda$ and the theorems of Plemmons–Cline [17] for the nonnegativity of the Moore–Penrose inverse can be similarly obtained.

Summary. This paper unifies all previously known results on nonnegative λ-monotone matrices where $1 \in \lambda$. The main theorem gives the structure of nonnegative matrices having a nonnegative {1}-inverse as a direct sum of certain "well-defined blocks" (of types (I) and (II)). One of the problems which has led to some of the interest in λ-monotone matrices is the problem of obtaining nonnegative solutions of a linear system $Ax = b$. As is well known, if X is a {1}-inverse (or {1, 3}-inverse) of A, then $x = Xb$ is a solution of $Ax = b$ if the system is consistent (or a best approximate solution), and so obviously Xb is nonnegative whenever X and b are both nonnegative. Of course, there is still much that remains to be done in order to characterize systems having nonnegative solutions. A recent paper by S. Friedland and H. Schneider [8] addresses this question.

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