

DECOMPOSITION OF NONNEGATIVE GROUP-MONOTONE MATRICES

BY

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ABSTRACT. A decomposition of nonnegative matrices with nonnegative group inverses has been obtained. This decomposition provides a new approach to the solution of problems relating to nonnegative matrices with nonnegative group inverses. As a consequence, a number of results are derived. Our results, among other things, answer a question of Berman, extend the theorems of Berman and Plemmons, DeMarr and Flor.

1. Introduction. Let A be an $m \times n$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, and (5) $AX = XA$ where X is an $n \times m$ real matrix and T denotes the transpose. For a rectangular matrix A and for a nonempty subset λ of $\{1, 2, 3, 4, 5\}$, X is called a λ -inverse of A if X satisfies equations (i) for each $i \in \lambda$. In particular, the $\{1, 2, 3, 4\}$ -inverse of A is the unique Moore-Penrose generalized inverse and is denoted by A^\dagger . A $\{1, 2\}$ -inverse of A which satisfies (5) is necessarily square and is called a group inverse. The group inverse of a matrix A , if it exists, is unique and is denoted by $A^\#$.

A matrix A is called group-monotone if $A^\#$ exists and is nonnegative. A matrix $A = (a_{ij})$ is called 0-symmetric if $a_{ij} = 0$ implies $a_{ji} = 0$. Thus every symmetric matrix and every positive matrix is 0-symmetric. A is called range-Hermitian (also called EPr) if the range of A is equal to the range of A^T , i.e., $R(A) = R(A^T)$. A is range-Hermitian if and only if $AA^\dagger = A^\dagger A$ and so $A^\dagger = A^\#$. An $m \times n$ matrix $A = (a_{ij})$ is called row (or column) stochastic if $a_{ij} \geq 0$ and $\sum_{j=1}^n a_{ij} = 1$, $1 \leq i \leq m$ (or $\sum_{i=1}^m a_{ij} = 1$, $1 \leq j \leq n$). If a matrix A is a direct sum of matrices S_i , then S_i will be called summands of A . A nonzero matrix A is called a zero divisor if $AB = 0$ or $BA = 0$ for some nonzero matrix B . For all other terminology the reader is referred to Ben-Israel and Greville [1].

Theorem 1 of this paper characterizes all nonnegative matrices A which have nonnegative group inverses; equivalently, $A^{(1,2)} = p(A) > 0$, where $p(A)$ is a polynomial in A with scalar coefficients. This theorem generalizes the known results for nonnegative matrices A whose A^\dagger is A [2] or, more generally, A^\dagger is some polynomial in A [7]. The solution to the problem raised by Berman of the characterization of all nonnegative matrices which are equal to a $\{1\}$ - or $\{1, 2\}$ -inverse of themselves also comes as a special case of Theorem 1. As a consequence of Theorem 1, we show (Corollary 2) that if A is a nonnegative matrix with $A^m = A$, $m > 2$, then $A = A_1 + A_2 + \cdots + A_k$, where $A_i > 0$; $A_i^m = A_i$; $A_i A_j = 0$, $i \neq j$;

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$d_i = \text{rank } A_i, d_i | m - 1$. This generalizes the theorem of DeMarr [5] for nonnegative idempotent matrices. Corollary 3 (Corollary 4) of Theorem 2 shows that for a nonnegative range-Hermitian (row stochastic) matrix A with $A^\# > 0, A^\# = A^\dagger = HA^m = A^m H$ ($A^\# = A^\dagger = A^m$) where H is a diagonal matrix with all entries positive. Theorem 4 characterizes all nonnegative rank factorizations of nonnegative group-monotone matrices. Theorems of Berman and Plemmons [3, Theorem 2 and Theorem 3] are also consequences of the characterizations obtained in Theorem 4. Our results, among others, depend on the following theorems proved in [6] and [7].

THEOREM A ([6, THEOREM 2]). *If E is a nonnegative idempotent matrix of rank r with no row or column completely zero. Then there exists a permutation matrix P such that*

$$PEP^T = \begin{pmatrix} x_1 y_1^T & & 0 \\ & \ddots & \\ 0 & & x_r y_r^T \end{pmatrix}$$

where x_i, y_i are positive vectors with $y_i^T x_i = 1$. In particular, E is 0-symmetric.

THEOREM B ([7, REMARK 3]). *Let A be a nonnegative matrix and $p(A) = \alpha_1 A^{m_1} + \dots + \alpha_k A^{m_k}, \alpha_i \neq 0, m_i > 0$, such that $p(A) > 0, Ap(A)$ is 0-symmetric, $Ap(A)A = A$, and $\text{rank } A = \text{rank } p(A)$. Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following three types (not necessarily all)*

(I) βxy^T , where x and y are positive vectors with $y^T x = 1$, and β is some positive number satisfying $\sum_m \alpha_i \beta^{m_i+1} = 1$:

(II)

$$\begin{pmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & \dots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \beta_{d-1d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \dots & 0 \end{pmatrix}$$

where x_i and y_i are positive vectors of the same order with $y_i^T x_i = 1, x_i$ and $x_j, i \neq j$, are not necessarily of the same order, and $\beta_{12}, \beta_{23}, \dots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d | m_i + 1$ for some m_i such that the product $\beta_{12} \beta_{23} \dots \beta_{d1}$ is a common root of the following system of at most d equations in t :

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1,$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0, \quad k \in \{1, \dots, d-1\},$$

where the summation in each of the above equations runs over all those m_i for which $d|(m_i + 1 - k)$, $k = 0, 1, \dots, d - 1$, with the convention that if there is no m_i for which $d|(m_i + 1 - k)$, $k \in \{1, \dots, d - 1\}$, then the corresponding equation is absent.

(III) *A zero matrix.*

In particular, if all $\alpha_i > 0$ then β in type (I) and the product $\beta_{12}\beta_{23}\dots\beta_{d1}$ in type (II) are unique. Further, in this case the positive integer d , i.e. the rank of a matrix of type (II), must divide each $m_i + 1$.

The concept of 0-symmetry has played a crucial role in the development of this paper.

2. Main results. Let A be any $n \times n$ matrix. Let us group the indices $i = 1, 2, 3, \dots, n$ into four disjoint sets according to whether the i th row and the i th column of A are both zero, or the i th row is zero but the i th column is not, and so on. Then by simultaneously rearranging rows and columns, we can find a permutation matrix P such that

$$PAP^T = \begin{pmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the diagonal blocks are square matrices such that K and L have no zero rows in common, and K and M have no zero columns in common. It may be noted that in certain situations some of the blocks may be absent. For example, if A is row stochastic then

$$PAP^T = \begin{pmatrix} K & 0 \\ M & 0 \end{pmatrix}.$$

In view of the frequent use of the above representation of a matrix throughout this paper, we record it in the following lemma.

LEMMA 1. *Let A be a square matrix. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where K and L have no zero rows in common, and K and M have no zero columns in common.

4

S. K. JAIN, E. K. KWAK AND V. K. GOEL

THEOREM 1. *Let A be a nonnegative matrix and $A^{(1,2)} = p(A) > 0$, where $p(A) = A^{m_1} + \dots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i > 0$. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where C, D are some nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I) βxy^T , where x and y are positive vectors with $y^T x = 1$ and β is a positive root

$$\sum_{m_i} \alpha_i t^{m_i+1} = 1. \tag{6}$$

(II)

$$\begin{pmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \beta_{d-1d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where x_i and y_i are positive vectors of the same order with $y_i^T x_i = 1$; x_i and x_j , $i \neq j$, are not necessarily of the same order; and $\beta_{12}, \dots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d | m_i + 1$ for some m_i such that the product $\beta_{12} \beta_{23} \dots \beta_{d1}$ is a common root of the following system of at most d equations in t

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1, \tag{7}$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0, \quad k \in \{1, 2, \dots, d-1\}, \tag{8}$$

where the summation in each of the above equations runs over all those m_i for which $m_i + 1 - k, k = 0, 1, 2, \dots, d - 1$, with the convention that if there is no m_i for which $d | (m_i + 1 - k)$, $k \in \{1, \dots, d - 1\}$, then the corresponding equation is absent.

In particular, if all $\alpha_i > 0$ then β in type (I) and the product $\beta_{12} \beta_{23} \dots \beta_{d1}$ in type (II) are unique. Further, in this case the positive integer d , i.e. the rank of a matrix of type (II), must divide each $m_i + 1$.

Conversely, if for some permutation matrix P

$$PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Let C, D be arbitrary nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

- (i) $\beta xy^T, \beta > 0, x, y$ are positive vectors with $y^T x = 1$.
- (ii)

$$\begin{pmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \beta_{d-1d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Let $\beta_{ij} > 0, x_i$ and y_i are positive vectors with $y_i^T x_i = 1$, then $A^{(1,2)} > 0$ and is similar to some polynomial in A with scalar coefficients.

PROOF. By Lemma 1, there exists a permutation matrix P_1 such that

$$P_1 A P_1^T = \begin{pmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $K, L, M,$ and N are nonnegative matrices such that K and L have no zero rows in common, and K and M have no zero columns in common. Since $A^2 = p(A)$, we have $A^2 p(A) = A$ and $A(p(A))^2 = p(A)$. From $A^2 p(A) = A$ we obtain $K^2 p(K) = K, Kp(K)L = L, MKp(K) = M,$ and $Mp(K)L = N$. Hence (i) is a nonnegative idempotent matrix. Since $Kp(K)K = K$ and $Kp(K)L = L$ and K has no zero rows in common, $Kp(K)$ cannot have a zero row. Similarly, no column of $Kp(K)$ is zero. Thus by Theorem A, $Kp(K)$ is 0-symmetric. Similarly, $A(p(A))^2 = p(A)$, we obtain $K(p(K))^2 = p(K)$, which, together with $K^2 p(K)$ gives $\text{rank } K = \text{rank } p(K)$. But then by Theorem B, there exists a permutation matrix P_2 such that $P_2 K P_2^T$ is a direct sum of matrices of types (I) or (II) (not necessarily both). Set

$$P = \begin{pmatrix} P_2 & 0 \\ 0 & I \end{pmatrix} P_1,$$

where the block matrices are of suitable orders. Then PAP^T is of the desired form. The converse is trivial.

ROLLARY 1. Let A be a nonnegative matrix of rank r and let $A^{(1,2)} = p(A) > 0, p(A) = \alpha_1 A^{m_1} + \dots + \alpha_k A^{m_k}, \alpha_i \neq 0, m_i > 0$. Then $A = A_1 + A_2 + \dots + A_k, A_i > 0; A_i A_j = 0, i \neq j; A_i^{(1,2)} = p(A_i); \text{rank } A_i = d_i, \sum_{i=1}^k d_i = r,$ and $d_i \leq m_i + 1$.

PROOF. By Theorem 1, there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where J is a direct sum of the matrices of the types (I) or (II) (not necessarily both) and C, D are nonnegative matrices of appropriate orders. Thus

$$J = \begin{bmatrix} S_1 & & & & 0 \\ & \ddots & & & \\ & & S_i & & \\ & & & \ddots & \\ 0 & & & & S_k \\ & & & & & 0 \end{bmatrix},$$

where S_i 's are of the types (I) or (II) and $\text{rank } S_i = d_i$. Set

$$J_i = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & S_i & \\ & & & & 0 \\ 0 & & & & & 0 \end{bmatrix}.$$

Then $A = \sum_{i=1}^k A_i$, where

$$A_i = P^T \begin{bmatrix} J_i & J_i D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C J_i & C J_i D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$

It can be easily verified that $A_i \geq 0$; $A_i A_j = 0, i \neq j$; $A_i^{(1,2)} = p(A_i)$; $\text{rank } A_i = d_i$, $\sum_{i=1}^k d_i = r$, and d_i divides some $m_j + 1$.

The corollary which follows generalizes DeMarr's theorem for nonnegative idempotent matrices [5].

COROLLARY 2. *Let A be a nonnegative matrix of rank r and let $A = A^m$, where $m > 2$ is a positive integer. Then $A = A_1 + A_2 + \dots + A_k$, where $A_i \geq 0$; $A_i A_j = 0, i \neq j$; $A_i^m = A_i$; $\text{rank } A_i = d_i, d_i | m - 1, \sum_{i=1}^k d_i = r$.*

PROOF. Follows from Corollary 1.

REMARK 1. Theorem 1 provides a complete solution, in a more general case, to the problem raised by Berman of characterization of the class of nonnegative matrices A with $\{1\}$ -inverse or $\{1, 2\}$ -inverse equal to A itself [2, Remark 5].

Henceforth by matrices of types (I) or (II), we will mean the matrices of types (I) or (II) described in Theorem 1.

THEOREM 2. *Let A be a nonnegative matrix having a nonnegative group inverse $A^\#$. Then $A^\# = K_1 A^m = A^m K_2$, where*

$$K_1 = P^T \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P, \quad K_2 = P^T \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P,$$

P is a permutation matrix, K is a diagonal matrix with positive diagonal entries, C, D are some nonnegative matrices of appropriate sizes, and m is a positive integer. Indeed, we may also choose

$$K_1 = K_2 = P^T \begin{pmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & CKD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P.$$

PROOF. By Theorem 1 there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where C, D are some nonnegative matrices of appropriate sizes and J is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if S is a summand of type (I) then $S^{(1,2)} = \beta^{-1}xy^T = \beta^{-2}S = \dots = \beta^{-k-1}S^k$, for any positive integer k . We show that if S is a summand of type (II) then $S^{(1,2)} = (\beta_{12}\beta_{23}\dots\beta_{d_1})^{-k}S^{kd-1}$ for any positive integer k . A straightforward verification shows that

$$S^{(1,2)} = \begin{pmatrix} 0 & 0 & & 0 & \frac{1}{\beta_{d_1}}x_1y_d^T \\ \frac{1}{\beta_{12}}x_2y_1^T & & \vdots & & 0 \\ 0 & \frac{1}{\beta_{23}}x_3y_2^T & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & & \frac{1}{\beta_{d-1d}}x_dy_{d-1}^T & 0 \end{pmatrix} \\ = (\beta_{12}\beta_{23}\dots\beta_{d_1})^{-1}S^{d-1}.$$

Also

$$S^{kd} = (\beta_{12}\beta_{23}\dots\beta_{d_1})^k \begin{pmatrix} x_1y_1^T & & 0 \\ & \ddots & \\ 0 & & x_dy_d^T \end{pmatrix}$$

for any positive integer k . Thus $S^{(1,2)} = (\beta_{12}\beta_{23}\dots\beta_{d_1})^{-k}S^{kd-1}$ as asserted. Now if $S_{11}, S_{12}, \dots, S_{1r}$ are summands of type (I) and $S_{21}, S_{22}, \dots, S_{2s}$ are summands of type (II) of ranks d_{21}, \dots, d_{2s} , respectively, then

$$\begin{aligned}
 J^{(1,2)} &= \begin{bmatrix} S_{11}^{(1,2)} & & & & 0 \\ & \ddots & & & \\ & & S_{1r}^{(1,2)} & & \\ & & & S_{21}^{(1,2)} & \\ & 0 & & & \ddots \\ & & & & & S_{2s}^{(1,2)} \end{bmatrix} \\
 &= \begin{bmatrix} \alpha_{11}I & & & & 0 \\ & \ddots & & & \\ & & \alpha_{1r}I & & \\ & & & \alpha_{21}I & \\ & 0 & & & \ddots \\ & & & & & \alpha_{2s}I \end{bmatrix} \\
 &\quad \cdot \begin{bmatrix} S_{11} & & & & 0 \\ & \ddots & & & \\ & & S_{1r} & & \\ & & & S_{21} & \\ & 0 & & & \ddots \\ & & & & & S_{2s} \end{bmatrix}^{(d_{21}d_{22}\cdots d_{2s})-1} \\
 &= \begin{bmatrix} S_{11} & & & & 0 \\ & \ddots & & & \\ & & S_{1r} & & \\ & & & S_{21} & \\ & 0 & & & \ddots \\ & & & & & S_{2s} \end{bmatrix}^{(d_{21}d_{22}\cdots d_{2s})-1} \\
 &\quad \cdot \begin{bmatrix} \alpha_{11}I & & & & 0 \\ & \ddots & & & \\ & & \alpha_{1r}I & & \\ & & & \alpha_{21}I & \\ & 0 & & & \ddots \\ & & & & & \alpha_{2s}I \end{bmatrix},
 \end{aligned}$$

where $\alpha_{ij}I$ are scalar matrices of appropriate sizes, $\alpha_{ij} > 0$. Thus $J^{(1,2)} = KJ^m = J^mK$, where

$$K = \begin{pmatrix} \alpha_{11}I & & & & 0 \\ & \ddots & & & \\ & & \alpha_{1r}I & & \\ & & & \alpha_{21}I & \\ 0 & & & & \ddots & \\ & & & & & \alpha_{2s}I \end{pmatrix}, \quad m = (d_{21} \cdots d_{2s}) - 1.$$

Thus

$$\begin{aligned} PA^{(1,2)}P^T &= \begin{pmatrix} KJ^m & KJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CKJ^m & CKJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J^m & J^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^m & CJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} J^m & J^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^m & CJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$A^{(1,2)} = P^T \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} PA^m = A^m P^T \begin{pmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P.$$

COROLLARY 3. *Under the hypothesis of Theorem 2, if A is also range-Hermitian, that is, $A^\# = A^\dagger$, then $A^\dagger = A^\# = HA^m = A^mH$ where H is a diagonal matrix with positive diagonal entries and m is a positive integer.*

PROOF. Let A be range-Hermitian. Then $A^\dagger = A^\#$. By Theorem 1,

$$PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where J is a direct sum of matrices of types (I) or (II). But then by using $A^\dagger = A^\#$, we obtain that C and D must be zero. Thus

$$PAP^T = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned}
 A^{\#} &= P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P A^m = A^m P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \\
 &= P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} P = P^T \begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \\
 &= P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} P = P^T \begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \\
 &= P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P A^m = A^m P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \\
 &= H A^m = A^m H,
 \end{aligned}$$

where H is a diagonal matrix with positive diagonal entries.

Before we prove Corollary 4 we record below a simple fact which we state without proof.

SUBLEMMA. *If βxy^T is a row (or column) stochastic matrix where $\beta > 0$ and x, y are positive vectors such that $y^T x = 1$ then $\beta = 1$.*

COROLLARY 4. *If A is an $n \times n$ nonnegative row (or column) stochastic matrix such that $A^{(1,2)} = p(A) > 0$, $p(A)$ is a polynomial in A with scalar coefficients, then $A^{(1,2)} = A^m$ for some positive integer m and $A^{(1,2)}$ is row (or column) stochastic.*

PROOF. For definiteness, let us assume that A is row stochastic. By appropriate application of Lemma 1 and Theorem 1, we can find a permutation matrix P such that

$$P A P^T = \begin{pmatrix} J & 0 \\ C J & 0 \end{pmatrix},$$

where J is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if S is a summand of type (I) then by sublemma $\beta = 1$ and so $S^{(1,2)} = xy^T = S$. Next, let S be a summand of type (II). Then S is a stochastic matrix and

$$S = \begin{pmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $\beta_{ij} > 0$ and x_i, y_i are positive vectors with $y_i^T x_i = 1$. A straightforward verification shows that $S^{(1,2)} = (\beta_{12} \beta_{23} \cdots \beta_{d1})^{-1} S^{d-1}$ and S^{d-1} is a row stochastic matrix. Thus again by the sublemma we get $(\beta_{12} \beta_{23} \cdots \beta_{d1}) = 1$. Then as in the proof of Theorem 2, we get $J^{(1,2)} = J^m$, and hence

$$A^{(1,2)} = P^T \begin{pmatrix} J^m & 0 \\ C J^m & 0 \end{pmatrix} P = A^m.$$

THEOREM 3. Every nonnegative rank factorization of nonnegative matrices J which are direct sum of matrices of types (I') or (II') (not necessarily both) is of the form $J = (FQ)(Q^TG)$, where Q is a permutation matrix, F and G are respectively the direct sum of matrices of the form (1) or (2) and (1') or (2'):

- (1) $\beta'x$,
- (1') $\beta''y^T$,
- (2)

$$\begin{pmatrix} 0 & \gamma_1 x_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_2 x_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \gamma_{d-1} x_{d-1} \\ \gamma_d x_d & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

(2')

$$\begin{pmatrix} \gamma'_1 y_1^T & & & & 0 \\ & \gamma'_2 y_2^T & & & \\ & & \dots & & \\ 0 & & & & \gamma'_d y_d^T \end{pmatrix}$$

such that $\beta' > 0$, $\beta'' > 0$, $\gamma_i > 0$, $\gamma'_i > 0$, x, y, x_i, y_i are positive vectors with $y^T x = 1$ and $y_i^T x_i = 1$. Moreover, $J^* = p(J)$ where $p(t) = \sum_{i=1}^k \alpha_i t^{m_i}$, $\alpha_i \neq 0$, $m_i > 0$, is some polynomial in t , $\beta' \beta''$ is a root of equation (6), and the product $\gamma_1 \gamma_2 \dots \gamma_d \gamma'_1 \gamma'_2 \dots \gamma'_d$ is a common root of the system of at most d equations (7) and (8). It is understood that in forming the product $(FQ)(Q^TG)$ if F has a summand of type (1) or (2) in the i th place of its direct sum then G has a corresponding summand of type (1') or (2') at the same i th place.

Also, for each nonnegative rank factorization $J = FG$ of J , $(GF)^{-1} = p(GF)$.

PROOF. Let $S = \beta xy^T$ with $y^T x = 1$ be a summand of J of type (I'). Clearly, the only possible nonnegative rank factorization of S is $S = FG$, $F = \beta'x$, $G = \beta''y^T$ with $\beta' \beta'' = \beta$. This gives $GF = \beta y^T x = \beta$. By equation (6), it then follows that $(GF)^{-1} = p(GF)$.

Next let

$$S = \begin{pmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \beta_{d-1} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

be a summand of J of type (II') of rank d . Let $S = FG$ be a nonnegative rank factorization of S . Partition $F = (F_{ij})$ and $G = (G_{ij})$ into matrix blocks such that for all i, j, k , $F_{ij}G_{jk}$ is defined and is of the same order as that of the (i, k) th block entry in S . Since F is of full column rank, no column of F is zero. Also no row of F can be zero. For otherwise, $S = FG$ shall have a zero row which is not true. Similarly, no row or column of G is zero. Since no column of F is zero, for each j there exists an i (depending on j) such that $F_{ij} \neq 0$. But then $G_{jk} = 0$ for all $k \neq i + 1$. Thus each row of the partitioned matrix G has at most one (and hence exactly one) nonzero entry. Clearly, then each column of G has also exactly one nonzero entry. The same reason yields that the partitioned matrix F has exactly one nonzero entry in each row and in each column. This implies there exists a permutation $\sigma \in S_d$ such that $F_{ij} = 0$, for all $j \neq \sigma(i)$, $G_{\sigma(i)k} = 0$, for all $k \neq i + 1$ and $F_{i\sigma(i)}G_{\sigma(i),i+1} = \beta_{i,i+1}x_i y_{i+1}^T$. But then the only solutions for $F_{i\sigma(i)}$ and $G_{\sigma(i),i+1}$ are given by $F_{i\sigma(i)} = \gamma_i x_i$ and $G_{\sigma(i),i+1} = \gamma'_i y_{i+1}^T$ where $\gamma_i \gamma'_i = \beta_{i,i+1}$. It is now clear that $J = (FQ)(Q^T G)$ where F and G are, respectively, the direct sum of matrices of the form (1) or (2) and (1') or (2') and Q is some permutation matrix. For summand of J of type (I') we have already shown that $(GF)^{-1} = p(GF)$. We now show the same for summand of type (II'). It is sufficient to prove the result for $S = FG$ where

$$F = \begin{pmatrix} 0 & \gamma_1 x_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_2 x_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \gamma_{d-1} x_{d-1} \\ \gamma_d x_d & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$G = \begin{pmatrix} \gamma'_1 y_1^T & & & & & 0 \\ & \gamma'_2 y_2^T & & & & \\ & & \dots & & & \\ 0 & & & & & \gamma'_d y_d^T \end{pmatrix}$$

Then

$$GF = \begin{pmatrix} 0 & \gamma_1 \gamma'_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_2 \gamma'_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \gamma_{d-1} \gamma'_{d-1} \\ \gamma_d \gamma'_d & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and $(GF)^d = (\gamma_1 \gamma_2 \dots \gamma_d \gamma'_1 \gamma'_2 \dots \gamma'_d)I$.

From equations (7) and (8), it follows that $(\gamma_1 \gamma_2 \dots \gamma_d \gamma'_1 \gamma'_2 \dots \gamma'_d)^{1/d}$ is a root of $\sum_{i=1}^k \alpha_i t^{m_i+1} = 1$. Thus $\sum_{i=1}^k \alpha_i (GF)^{m_i+1} = I$, since $(GF)^d = (\gamma_1 \gamma_2 \dots \gamma_d \gamma'_1 \gamma'_2 \dots \gamma'_d)I$. Hence $(GF)p(GF) = I$, completing the proof.

The next theorem describes all nonnegative rank factorizations of a nonnegative matrix with a nonnegative group inverse.

THEOREM 4. (a) Let $A \geq 0$ and P be a permutation matrix such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

J is a direct sum of matrices of types (I') or (II'), C, D are some nonnegative matrices of suitable sizes (equivalently, $A \geq 0$ and $A^\# = p(A) \geq 0$). Then we have the following:

(a₁) If $J = FG$ is a nonnegative rank factorization of J then it "lifts" to a nonnegative rank factorization

$$A = P^T \begin{bmatrix} F \\ 0 \\ CF \\ 0 \end{bmatrix} (G \quad GD \quad 0 \quad 0)P$$

of A .

(a₂) If $A = F'G'$ is a nonnegative rank factorization of A then it "contracts" to a nonnegative rank factorization $J = F'_{11}G'_{11}$ of J where F'_{11} (G'_{11}) consists of first n rows (columns) of PF' ($G'P^T$), n being the order of the matrix J .

(a₃) If σ denotes the operation of "lifting" defined in (a₁), and η denotes the operation of "contracting" defined in (a₂) then $\sigma\eta = \text{identity} = \eta\sigma$. Thus there is a 1-1 correspondence between the class of nonnegative rank factorizations of J and the class of nonnegative rank factorizations of A .

(a₄) If $A = F'G'$ and $J = FG$ are corresponding nonnegative rank factorizations of A and J , respectively, then $GF = G'F'$ and $(GF)^{-1} = p(GF)$, where $A^\# = p(A)$.

(b) If $A = FG$ is a nonnegative rank factorization of A such that $(GF)^{-1} = p(GF)$ where $p(t)$ is some polynomial in t with scalar coefficients then $A^\#$ exists, $A^\# \geq 0$, and $A^\# = p(A)$.

PROOF. (a₁) Straightforward verification.

(a₂) Let n be the order of J . Partition

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad PF' = \begin{pmatrix} F'_{11} \\ F'_{21} \end{pmatrix}, \quad \text{and} \quad G'P^T = (G'_{11}G'_{12})$$

where F'_{11} (G'_{11}) consists of first n rows (columns) of PF' ($G'P^T$) respectively. By comparing we get $J = F'_{11}G'_{11}$ which is clearly nonnegative rank factorization of J .

(a₃) It is obvious that if we perform the operation of lifting followed by the operation of contracting then the composition is identity operation, i.e. $\eta\sigma = \text{identity}$. On the other hand, it is not clear to us that $\sigma\eta$ is also identity, in general. However, we can show $\sigma\eta = \text{identity}$ under our hypothesis. Let $A = F'G'$ be a nonnegative rank factorization of A . By performing the operation η we get

$J = F'_{11}G'_{11}$ where F'_{11} and G'_{11} are respectively the first n rows and the first n columns of PF' and $G'P^T$. Then by performing σ , we get

$$A = P^T \begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix} (G'_{11} \ G'_{11}D \ 0 \ 0)P.$$

To prove $\sigma\eta = \text{identity}$, we need to show

$$PF' = \begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix}, \quad G'P^T = (G'_{11} \ G'_{11}D \ 0 \ 0). \tag{9}$$

Since by Theorem 3, $F'_{11} (G'_{11})$ is the direct sum of matrices of the form (1) or (2) ((1') or (2')) it is sufficient to prove (9) when F'_{11} is of the form (1) or (2) and G'_{11} is of the corresponding form (1') or (2'). Partition

$$PF' = \begin{pmatrix} F'_{11} \\ F'_{21} \\ F'_{31} \\ F'_{41} \end{pmatrix}, \quad G'P^T = (G'_{11} \ G'_{12} \ G'_{13} \ G'_{14})$$

such that the size of F'_{j1} , $j = 1, 2, 3, 4$, is the same as that of the corresponding block in

$$\begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix}.$$

Similarly, for the size of G'_{1j} , $j = 1, 2, 3, 4$. Then comparing the corresponding blocks in

$$\begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix} (G'_{11} \ G'_{11}D \ 0 \ 0) = \begin{pmatrix} F'_{11} \\ F'_{21} \\ F'_{31} \\ F'_{41} \end{pmatrix} (G'_{11} \ G'_{12} \ G'_{13} \ G'_{14})$$

we get the result by observing that F'_{11} and G'_{11} are not zero divisors.

(a₄) Follows from (a₁)–(a₃) and Theorem 3.

(b) Recall that if $A = FG$ is a rank factorization of A then $A^\#$ exists if and only if $(GF)^{-1}$ exists [4]. In this case $A^\# = F(GF)^{-2}G$. A straightforward computation then yields $A^\# = p(A)$ where $(GF)^{-1} = p(GF)$.

REMARK 2. Theorem 3 along with Theorem 4(a₃) characterizes all the nonnegative rank factorizations of a nonnegative matrix with nonnegative group inverse.

REMARK 3. Another proof of Theorem 4(a₄) can be given on the same lines as the proof for the special case when $p(A) = A$ given by Berman and Plemmons [3]. However, the purpose of Theorem 4 is to characterize all nonnegative rank factorizations of nonnegative matrices A with $A^* > 0$, and (a₄) comes out as an offshoot.

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